

TITS INDICES OVER SEMILOCAL RINGS

V. PETROV AND A. STAVROVA

ABSTRACT. We give a simplified proof of Tits' classification of semisimple algebraic groups that remains valid over semilocal rings. In particular, we provide explicit necessary and sufficient conditions that anisotropic groups of a given type appear as anisotropic kernels of semisimple groups of a given Tits index. We also give a new proof of the existence of all indices of exceptional inner type using the notion of canonical dimension of projective homogeneous varieties.

1. INTRODUCTION

In his famous paper [11] Jacques Tits showed that any semisimple group G over a field is determined by its anisotropic kernel and a combinatorial datum called the *Tits index* of G . Some arguments were sketched or omitted there, and appeared in later papers. Namely, Selbach [9] clarified the proof of the completeness of the list of Tits indices, and in [13] Tits himself has finished the proof of the existence of all indices; see also [10] and [14, Appendix] for more detailed expositions.

The goal of the present paper is to show that the Tits classification carries over to arbitrary connected semilocal rings. We do not make use of the field case, but rather provide a shortened and simplified version of Tits' arguments. We also give a new proof of the existence of all indices of inner exceptional type using the notion of canonical dimension of projective homogeneous varieties of semisimple algebraic groups.

Our proof of Tits classification consists of two parts: combinatorial and representation-theoretic. Combinatorial restrictions follow from the presence of the opposition involution on the extended Dynkin diagram. These restrictions allow to exclude most of the “wrong” indices (Proposition 3). Representation-theoretic arguments allow to define Tits algebras of a semisimple group in the same fashion as this was done by Tits [12] over fields (Theorem 1). In Theorem 2 we give a necessary and sufficient condition in terms of Tits algebras that a semisimple group scheme H can be embedded into a larger semisimple group G as the derived subgroup of a Levi subgroup of a fixed parabolic subgroup of G . Combining this result with the combinatorial restrictions, we obtain the list of all possible indices, and show that the existence of a group with a given index is equivalent to the existence of an anisotropic group (its anisotropic kernel) subject to certain explicitly stated conditions (§ 6, Theorem 3). In the field case, these conditions appeared earlier in [9, 13, 4, 14] for all Tits indices except those where the $*$ -action of the Galois group on the set of “circled” vertices is non-trivial.

Our proof of the existence of all indices of inner exceptional type (Theorem 4) is based on the knowledge of the list of maximal possible values of the p -relative

The first author is partially supported by PIMS fellowship, RFBR 08-01-00756, RFBR 09-01-90304 and RFBR 09-01-91333. Both authors are supported by RFBR 09-01-00878.

canonical dimension of the variety of Borel subgroups of simple algebraic groups corresponding to generic torsors. This list was obtained in [8] by means of the J-invariant of algebraic groups.

2. SEMISIMPLE GROUP SCHEMES

In this section we reproduce some definitions and results of [2]. Throughout the paper, all references starting with Exp. YZ are to this source.

Let S be a scheme (not necessarily separated). A group scheme G over S is called *reductive* if it is affine and smooth over S , and its geometric fibers $G_{\overline{k(s)}}$ are connected reductive groups in the usual sense for all $s \in S$ (Exp. XIX Déf. 4.7). When S is reduced, the smoothness can be replaced by the condition that G is finitely presented over S and the dimension of a fiber is locally constant (see Exp. VI_B, Cor. 4.4). The *type* of G at $s \in S$ is the root datum of $G_{\overline{k(s)}}$. The type is locally constant (Exp. XXII Prop. 2.8). To simplify the exposition, in the sequel we consider reductive group schemes of constant type only. Thus the type of a reductive group scheme G is a root datum $\mathcal{R} = (\Phi, \Lambda, \Phi^*, \Lambda^*)$, where Φ is a root system, called the *root system* of G , Λ is a \mathbb{Z} -lattice containing Φ , called the *lattice of weights* of G , and Φ^* and Λ^* are the dual objects (Exp. XXI Déf. 1.1.1). A reductive group G is *semisimple*, if the rank of Φ equals that of Λ . We also usually include in the type a fixed subset of positive roots Φ^+ in Φ , which determines a system of simple roots of Φ and, therefore, a Dynkin diagram D .

Over any scheme S there exists a unique *split* group scheme G_0 of a given type \mathcal{R} , which actually comes from a group scheme over $\text{Spec } \mathbb{Z}$ known as the Chevalley – Demazure group scheme (Exp. XXV Thm. 1.1). *Quasi-split* group schemes over S of the same type as G_0 are parametrized by $H^1(S, \text{Aut}(\mathcal{R}, \Phi^+))$, where $\text{Aut}(\mathcal{R}, \Phi^+)$ is the group of automorphisms of \mathcal{R} preserving Φ^+ (cf. Exp. XXIV Thm. 3.11). All cohomology groups we consider are with respect to the fpqc topology (but note that $H^1(S, H) = H_{\text{ét}}^1(S, H)$ when H is smooth).

Every semisimple group scheme G is an inner twisted form of a uniquely determined quasi-split group G_{qs} , given by a cocycle $\xi \in Z^1(S, G_{qs}^{\text{ad}})$, where G_{qs}^{ad} is the adjoint group acting on G_{qs} by inner automorphisms. Cocycles in the same class in $H^1(S, G_{qs}^{\text{ad}})$ produce isomorphic group schemes (Exp. XXIV 3.12.1); however, distinct classes of cocycles may correspond to the same isomorphism class of groups (see below).

A Dynkin diagram D is nothing but a finite set of vertices together with a subset $E \subseteq D \times D$ of edges and a length function $D \rightarrow \{1, 2, 3\}$ (in other words, a colored graph). The scheme-theoretic counterpart of this notion is called a *Dynkin scheme* (Exp. XXIV § 3). So a Dynkin scheme over S is a twisted finite scheme \mathcal{D} over S together with a subscheme $\mathcal{E} \subseteq \mathcal{D} \times_S \mathcal{D}$ and a map $\mathcal{D} \rightarrow \{1, 2, 3\}_S$. Isomorphisms, base extensions and constant Dynkin schemes are defined in an obvious way. We denote by D_S the constant Dynkin scheme over S corresponding to a Dynkin diagram D . By $\text{Aut}(\mathcal{D})$ we always mean the scheme of automorphisms of \mathcal{D} over S as a Dynkin scheme; it is a twisted constant group scheme over S .

To any semisimple group scheme G one associates the Dynkin scheme $\text{Dyn}(G)$. For a quasi-split group $\text{Dyn}(G_{qs})$ is a twisted form of D_S corresponding to the image in $Z^1(S, \text{Aut}(D))$ of a cocycle $\xi \in Z^1(S, \text{Aut}(\mathcal{R}, \Phi^+))$ defining G_{qs} under the map

induced by the canonical map $\text{Aut}(\mathcal{R}, \Phi^+) \rightarrow \text{Aut}(D)$ (Exp. XXIV 3.7). When G_{qs} is simply connected or adjoint, the latter map is an isomorphism.

In general, $\text{Dyn}(G)$ is isomorphic to $\text{Dyn}(G_{qs})$, but the isomorphism is not canonical. By an *orientation* u on G we mean a choice of an element $u \in \text{Isomext}(G_{qs}, G)(S)$, that is of an isomorphism between $\text{Dyn}(G_{qs})$ and $\text{Dyn}(G)$. A notion of an isomorphism of oriented group schemes is defined obviously. Exp. XXIV Rem. 1.11 shows that $H^1(S, G_{qs}^{ad})$ is in bijective correspondence with the set of isomorphism classes of *oriented* inner twisted forms of G_{qs}^{ad} .

Let T/S be a Galois covering that splits $\text{Dyn}(G)$, i.e. $\text{Dyn}(G)_T \simeq D_T$. For example, one can take as T the torsor corresponding to the cocycle in $Z^1(S, \text{Aut}(D))$. Every element $\sigma \in \text{Aut}(T/S)$ acts on $\text{Dyn}(G)_T$ and therefore defines some $\varphi_\sigma \in \text{Aut}(D)(T)$ such that the diagram

$$\begin{array}{ccc} D_T & \xrightarrow{\varphi_\sigma} & D_T \\ \downarrow & & \downarrow \\ T & \xrightarrow{\sigma} & T \end{array}$$

commutes. By Galois descent this action (which is called *the $*$ -action*) completely determines $\text{Dyn}(G)$. If S is connected, the $*$ -action can be considered as an action of $\text{Aut}(T/S)$ on the Dynkin diagram D , and extends by \mathbb{Q} -linearity to the $*$ -action on Λ .

A subgroup scheme P of G is called *parabolic* if it is smooth and $\overline{P_{k(s)}}$ is a parabolic subgroup of $G_{\overline{k(s)}}$ in the usual sense for every $s \in S$ (Exp. XXVI Déf. 1.1). To a parabolic subgroup P one can attach the *type* $\mathfrak{t}(P)$ of P which is a clopen subscheme of $\text{Dyn}(G)$ (Exp. XXVI 3.2). Note that the clopen subschemes of $\text{Dyn}(G)$ are in one-to-one correspondence with the $*$ -invariant clopen subschemes of D_T , where T/S is as above.

If L is a Levi part of P , we have a canonical map $\text{Dyn}(L) \rightarrow \text{Dyn}(G)$ depending only on L and G . In particular, an orientation on G *induces* an orientation on L .

3. REPRESENTATION-THEORETIC LEMMAS

By a *representation* of a group scheme G over S we mean a homomorphism of algebraic groups $\rho: G \rightarrow \text{GL}_1(A)$, where A is an Azumaya algebra (more formally, a sheaf of Azumaya algebras) over S .

Let G_0 be a split semisimple group scheme over a scheme S , and let $G_0 \rightarrow \text{GL}(V)$ be a representation of G_0 on a projective module (more formally, a locally free sheaf of modules) V of finite rank over S . Fix a maximal split torus T_0 of G_0 and let Λ and Λ_r be its lattices of weights and roots respectively. Then V decomposes into a direct sum $\bigoplus_{\lambda \in \Lambda} V_\lambda$ so that for any scheme S' over S , any $t \in T_0(S')$, and any $v \in V_\lambda(S')$ one has $\rho(t)v = \lambda(t)v$ (Exp. I Prop. 4.7.3). A character λ with $V_\lambda \neq 0$ is called a *weight* of V .

The *cocenter* $\text{Cocent}(G)$ of G is the group scheme $\text{Hom}(\text{Cent}(G), \mathbf{G}_m)$. When G is split it can be identified with the constant group scheme $(\Lambda / \Lambda_r)_S$. Descent shows that $\text{Cent}(G)$ is isomorphic to $\text{Cent}(G_{qs})$, and therefore $\text{Cocent}(G)$ is isomorphic to $\text{Cocent}(G_{qs})$. The isomorphism depends only on the orientation of G .

A representation $\rho: G \rightarrow \text{GL}_1(A)$ will be called *center preserving* if $\rho(\text{Cent}(G)) \subseteq \text{Cent}(\text{GL}_1(A))$. In this case ρ induces a homomorphism $\rho^{ad}: G^{ad} \rightarrow \text{PGL}_1(A)$ and

determines an element $\lambda_\rho \in \text{Cocent}(G)(S)$, which is the restriction of ρ to $\text{Cent}(G)$ composed with the natural isomorphism $\text{Cent}(\text{GL}_1(A)) \simeq \mathbf{G}_m$.

- Lemma 1.** (1) $G \rightarrow \text{GL}(V)$ is center preserving if and only if over a splitting covering $\coprod S_\tau \rightarrow S$ every two weights of V differ by an element of Λ_r .
 (2) The dual $G \rightarrow \text{GL}(V^*)$ of a center preserving representation $G \rightarrow \text{GL}(V)$ is center preserving.
 (3) The tensor product $G \rightarrow \text{GL}(V_1 \otimes V_2)$ of center preserving representations $G \rightarrow \text{GL}(V_1)$ and $G \rightarrow \text{GL}(V_2)$ is center preserving.
 (4) For any representation $\rho: G \rightarrow \text{GL}(V)$ and an element $\lambda \in \text{Cocent}(G)(S)$, the submodule $W \subseteq V$ defined by

$$W(S') = \{v \in V \times_S S' \mid c \cdot v = \lambda(c)v \text{ for all fpqc } S''/S' \text{ and } c \in \text{Cent}(G)(S'')\}$$

is a G -invariant direct summand of V . Moreover, the representation $\rho': G \rightarrow \text{GL}(W)$ is center preserving and $\lambda_{\rho'} = \lambda$ if $W \neq 0$.

Proof. For (1) observe that since the condition $\rho(\text{Cent}(G)) \subseteq \text{Cent}(\text{GL}(V))$ is local with respect to fpqc topology, we can assume that G is split. Then V is center preserving if and only if restrictions of every two weights λ and μ of V to $\text{Cent}(G)$ coincide. This means exactly that $\lambda - \mu$ belongs to Λ_r (Exp. XXII Rem. 4.1.8). Parts (2) and (3) follow from (1).

To prove (4), define $W'(S')$ as the set of all $v \in V(S')$ such that there exist an fpqc covering $\coprod S'_\tau \rightarrow S'$ and, for each τ , a finite number of elements $\lambda_1, \dots, \lambda_k \in \text{Cocent}(G)(S'_\tau)$ distinct from λ and elements $v_1, \dots, v_k \in V \times_S S'_\tau$ such that $v = v_1 + \dots + v_k$ and $cv_i = \lambda_i(c)v_i$ for all fpqc S''_τ/S'_τ and $c \in \text{Cent}(G)(S''_\tau)$. Obviously W and W' are G -invariant (sheaves of) submodules of V . Over a splitting covering of G it is easily seen that $V = W \oplus W'$; therefore it is also true over the base S . By construction the representation $\rho': G_{qs} \rightarrow W$ is center preserving and $\lambda_{\rho'} = \lambda$. \square

Lemma 2. Let G_{qs} be a quasi-split group over S . Then any element of $\text{Cocent}(G_{qs})(S)$ appears as λ_ρ for some center preserving representation $\rho: G_{qs} \rightarrow \text{GL}(V)$.

Proof. Over a splitting covering of G_{qs} choose a weight $\lambda \in \Lambda$ that represents a given element of $\text{Cocent}(G_{qs})(S)$. Obviously $\lambda + \Lambda_r$ is $*$ -invariant. It is known (see [1, Ch. VI, Exerc. 5 du §2]) that any weight is equivalent modulo Λ_r to a minuscule weight. On the other hand, by [12, 3.1] we have $(\Lambda / \Lambda_r)^* = \Lambda^* / \Lambda_r^*$. So we may assume that λ is a $*$ -invariant minuscule weight.

Consider first the split group G_0 over \mathbb{Z} . Recall briefly the construction of a Weyl module $V(\lambda)$ for G_0 (see [5] for details). We start from a finite dimensional irreducible $(G_0)_{\mathbb{C}}$ -module with the highest weight λ ; we fix a vector v_+ of the weight λ (which is unique up to a scalar). Denote by \mathfrak{U} the universal enveloping algebra of the Lie algebra of $(G_0)_{\mathbb{C}}$, by \mathfrak{U}^+ and \mathfrak{U}^- its subalgebras generated by the positive (respectively, negative) root subspaces, and by $\mathfrak{U}_{\mathbb{Z}}, \mathfrak{U}_{\mathbb{Z}}^+, \mathfrak{U}_{\mathbb{Z}}^-$ their \mathbb{Z} -forms used in the Chevalley's construction of split reductive groups. Then $V(\lambda)$ is defined as $\mathfrak{U}_{\mathbb{Z}}^- v_+$. Note that $V(\lambda)$ is center preserving by Lemma 1, (1).

Let Γ be a subgroup of $\text{Aut}(\mathcal{R}, \Phi^+)$ preserving λ . Then any element $\gamma \in \Gamma$ induces an automorphism of $\mathfrak{U}_{\mathbb{Z}}$ which preserves $\mathfrak{U}_{\mathbb{Z}}^+$ and $\mathfrak{U}_{\mathbb{Z}}^-$. Since γ preserves λ , the representations $\rho: (G_0)_{\mathbb{C}} \rightarrow \text{GL}(V(\lambda)_{\mathbb{C}})$ and $\rho \circ \gamma: (G_0)_{\mathbb{C}} \rightarrow \text{GL}(V(\lambda)_{\mathbb{C}})$ are equivalent, and their differentials are equivalent as well. Therefore, there exists

$\varphi \in \mathrm{GL}(V(\lambda)_{\mathbb{C}})$ such that $\gamma(g)\varphi(v) = \varphi(gv)$ for every $v \in V(\lambda)_{\mathbb{C}}$ and $g \in \mathfrak{U}$; moreover, φ is unique up to a scalar. It is easy to see that φ preserves the line spanned by v_+ , and we can normalize φ so that $\varphi(v_+) = v_+$. Now,

$$\varphi(\mathfrak{U}_{\mathbb{Z}}^- v_+) \leq \gamma(\mathfrak{U}_{\mathbb{Z}}^-)\varphi(v_+) = \mathfrak{U}_{\mathbb{Z}}^- v_+,$$

so φ induces an automorphism $\varphi_{\mathbb{Z}}$ of $V(\lambda)$ compatible with γ and preserving v_+ . Since $\mathbb{Z}[G_0]$ is a Hopf subalgebra of $\mathbb{Q}[G_0]$ and $V(\lambda)$ is a subcomodule of $V(\lambda)_{\mathbb{Q}}$, and \mathbb{C}/\mathbb{Q} is faithfully flat, $\varphi_{\mathbb{Z}}$ is an equivalence of the representations $\rho: G_0 \rightarrow \mathrm{GL}(V(\lambda))$ and $\rho \circ \gamma: G_0 \rightarrow \mathrm{GL}(V(\lambda))$. Moreover, since $\varphi_{\mathbb{Z}}$ is uniquely determined by γ , we obtain a homomorphism $\psi: \Gamma \rightarrow \mathrm{GL}(V(\lambda))$.

Now let ξ be a cocycle in $Z^1(S, \Gamma)$ producing G_{qs} . The cocycle $\psi_*(\xi)$ then defines a projective module V together with a representation $G_{qs} \rightarrow \mathrm{GL}(V)$ we need. \square

4. TITS ALGEBRAS

Theorem 1. *Let (G, u) be an oriented semisimple group scheme of constant type over S corresponding to the class $[\xi] \in H^1(S, G_{qs}^{ad})$.*

- (1) *There exist two natural mutually quasi-inverse equivalences F_u, F'_u between the categories of group schemes over S with G_{qs}^{ad} -action (by group automorphisms) and group schemes over S with G^{ad} -action. In particular, each center preserving representation $\rho: G_{qs} \rightarrow \mathrm{GL}(V)$ gives rise to a center preserving representation $F_u(\rho): G \rightarrow \mathrm{GL}_1(A_{u,\rho})$ for some Azumaya algebra $A_{u,\rho}$.*
- (2) *The class $[A_{u,\rho}]$ in the Brauer group $\mathrm{Br}(S)$ depends only on $\lambda_{F_u(\rho)}$ and not on the particular choice of u and ρ . Its image in $H^2(S, \mathbf{G}_m)$ coincides with $(\lambda_\rho)_*\delta([\xi])$, where*

$$(\lambda_\rho)_*: H^2(S, \mathrm{Cent}(G_{qs})) \rightarrow H^2(S, \mathbf{G}_m),$$

and δ is the connecting homomorphism in the long exact sequence arising from the sequence

$$1 \longrightarrow \mathrm{Cent}(G_{qs}) \longrightarrow G_{qs} \longrightarrow G_{qs}^{ad} \longrightarrow 1.$$

Proof. 1. Consider the left G^{ad} - and right G_{qs}^{ad} -torsor $I = \mathrm{Isomint}_u(G_{qs}, G)$ (see Exp. XXIV Rem. 1.11). Let H be a group scheme with a G_{qs}^{ad} -action. Then $F_u(H) = I \times^{G_{qs}^{ad}} H$ is a group scheme over $I/G_{qs}^{ad} \simeq S$ with a left G^{ad} -action. Similarly, F'_u is defined by $F'_u(H') = I' \times^{G^{ad}} H'$, where $I' = \mathrm{Isomint}_{u^{-1}}(G, G_{qs})$. Further, we have isomorphisms $I' \times^{G^{ad}} I \simeq G_{qs}^{ad}$ and $I \times^{G_{qs}^{ad}} I' \simeq G^{ad}$, hence F_u and F'_u are mutually quasi-inverse.

2. The cohomological class in $H^1(S, \mathrm{PGL}(V))$ corresponding to $A_{u,\rho}$ is nothing but $\rho_*^{ad}([\xi])$, where $\rho_*^{ad}: G_{qs}^{ad} \rightarrow \mathrm{PGL}(V)$ is the representation induced by ρ . Now the last assertion of the Theorem follows from the commutativity of the diagram

$$\begin{array}{ccc} H^1(S, G_{qs}^{ad}) & \xrightarrow{\delta} & H^2(S, \mathrm{Cent}(G_{qs})) \\ \rho_*^{ad} \downarrow & & \downarrow (\lambda_\rho)_* \\ H^1(S, \mathrm{PGL}(V)) & \longrightarrow & H^2(S, \mathbf{G}_m), \end{array}$$

which comes from the diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \text{Cent}(G_{qs}) & \longrightarrow & G_{qs} & \longrightarrow & G_{qs}^{ad} \longrightarrow 1 \\
 & & \lambda_\rho \downarrow & & \rho \downarrow & & \rho^{ad} \downarrow \\
 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \text{GL}(V) & \longrightarrow & \text{PGL}(V) \longrightarrow 1.
 \end{array}$$

Thus, once u is fixed, the class of $A_{u,\rho}$ depends only on λ_ρ . Let v be another orientation on G . Then $\rho' = F'_v(F_u(\rho))$ is ρ composed with the corresponding outer automorphism of G_{qs} ; in particular, its target is still $\text{GL}(V)$. Obviously $F_v(\rho') \simeq F_u(\rho)$. Now, if σ is another representation of G_{qs} with $\lambda_{F_v(\sigma)} = \lambda_{F_u(\rho)}$, then

$$\lambda_\sigma = \lambda_{F_v(\sigma)} \circ v = \lambda_{F_u(\rho)} \circ v = \lambda_{F_v(\rho')} \circ v = \lambda_{\rho'},$$

hence

$$[A_{v,\sigma}] = [A_{v,\rho'}] = [A_{u,\rho}].$$

□

The Azumaya algebra $A_{u,\rho}$ will be called the *Tits algebra* of G corresponding to a center preserving representation $\rho: G_{qs} \rightarrow \text{GL}(V)$. We denote by β_G the homomorphism

$$\begin{aligned}
 \beta_G: \text{Cocent}(G)(S) &\rightarrow \text{Br}(S) \\
 \lambda &\mapsto [A_{u,\rho}] \text{ with } \lambda_{F_u(\rho)} = \lambda.
 \end{aligned}$$

It is well-defined in view of Lemma 2 and Theorem 1. To see that β_G is indeed a homomorphism one can use either the tensor product of representations or the fact that $\text{Br}(S)$ is a subgroup in $H^2(S, \mathbf{G}_m)$.

If the orientation u is fixed, we will consider β_G as a homomorphism from $\text{Cocent}(G_{qs})$ to $\text{Br}(S)$. Further, for an element λ of Λ^* we will write $\beta_G(\lambda)$ instead of $\beta_G(\lambda|_{\text{Cent}(G_{qs})})$.

The Dynkin scheme $\text{Dyn}(G)$ is the disjoint union of its minimal clopen subschemes which will be called *orbits* for brevity; they indeed correspond to orbits of the $*$ -action on a set of simple roots.

Assume that G is simply connected. Let T_{qs} be a fixed maximal torus of G_{qs} , T_{qs}^{ad} be the respective torus in G_{qs}^{ad} . Over a splitting covering we have two canonical homomorphisms

$$\begin{aligned}
 \omega: \text{Dyn}(G) &\rightarrow \text{Hom}(T_{qs}, \mathbf{G}_m), \\
 \alpha: \text{Dyn}(G) &\rightarrow \text{Hom}(T_{qs}^{ad}, \mathbf{G}_m),
 \end{aligned}$$

that associate to each vertex i of the Dynkin diagram the fundamental weight ω_i or, respectively, the simple root α_i . By faithfully flat descent these homomorphisms are defined over the base scheme S .

Let O be an orbit in $\text{Dyn}(G)$. Composing ω (resp., α) with the inclusion $O \rightarrow \text{Dyn}(G)$, we obtain a weight $\omega_O: (T_{qs})_O \rightarrow \mathbf{G}_m$ (resp., a root $\alpha_O: (T_{qs}^{ad})_O \rightarrow \mathbf{G}_m$), which will be called the *canonical weight* (resp., the *canonical root*) corresponding to O (cf. Exp. XXIV 3.8). It is easy to see that α_O and ω_O are $*$ -invariant weights of G_O .

Recall that the Weil restriction $R_{S'/S}$ ($\prod_{S'/S}$ in the notation of [2]) is the right adjoint to the base change functor. So we have homomorphisms

$$\begin{aligned}\bar{\omega}_O: T_{qs} &\rightarrow R_{O/S}(\mathbf{G}_m), \\ \bar{\alpha}_O: T_{qs}^{ad} &\rightarrow R_{O/S}(\mathbf{G}_m).\end{aligned}$$

If O splits over an extension S'/S into a disjoint union $\coprod_i O_i$, then $(\bar{\omega}_O)_{S'}$ (resp. $(\bar{\alpha}_O)_{S'}$) is equal to $\prod_i \omega_{O_i}$ (resp., $\prod_i \alpha_{O_i}$) composed with the natural isomorphism $\prod_i R_{O_i/S'}(\mathbf{G}_m) \simeq R_{\coprod_i O_i/S'}(\mathbf{G}_m)$. In particular, over a splitting covering $\bar{\omega}_O$ (resp. $\bar{\alpha}_O$) can be identified with an appropriate product of ω_i (resp., α_i).

Proposition 1. (1) *In the above setting we have the isomorphism*

$$\prod_O \bar{\omega}_O: T_{qs} \simeq \prod_O R_{O/S}(\mathbf{G}_m)$$

(cf. Exp. XXIV Prop. 3.13).

(2) *If L'_{qs} is the standard Levi subgroup of a standard parabolic subgroup P in G_{qs}^{ad} , then we have the isomorphism*

$$\prod_{O: O \not\subset \mathfrak{t}(P)} \bar{\alpha}_O: \text{Cent}(L'_{qs}) \simeq \prod_{O: O \not\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_m).$$

(3) *We have*

$$L'_{qs} = \text{Cent}_{G_{qs}}(Q) = \text{Cent}_{G_{qs}}(Q_{diag}),$$

where Q is the natural split subtorus $\prod_{O: O \not\subset \mathfrak{t}(P)} \mathbf{G}_m$ of $\prod_{O: O \not\subset \mathfrak{t}(P)} R_{O/S}(\mathbf{G}_m)$, and Q_{diag} is the split torus of rank 1 embedded diagonally into Q .

Proof. Let's prove (2). Note that $\text{Cent}(L'_{qs})$ is contained in T_{qs}^{ad} , so the map is well-defined. Over each element S_τ of a splitting covering of S the Dynkin scheme can be identified with a set D and $\mathfrak{t}(P)$ with a subset $D \setminus J$. The map $\prod_{O: O \not\subset \mathfrak{t}(P)} \bar{\alpha}_O$ becomes $\prod_{i \in J} \alpha_i$, and $\text{Cent}(L'_{qs})_{S_\tau}$ equals $\bigcap_{i \in D \setminus J} \text{Ker } \alpha_i$. But

$$\prod_{i \in D} \alpha_i: (T_{qs}^{ad})_{S_\tau} \rightarrow \prod_{i \in D} \mathbf{G}_m$$

is an isomorphism, and (2) follows. Part (1) can be proved similarly and even easier.

We have obvious inclusions

$$L'_{qs} \leq \text{Cent}_{G_{qs}}(Q) \leq \text{Cent}_{G_{qs}}(Q_{diag}),$$

so to prove (3) it suffices to show that $H = \text{Cent}_{G_{qs}}(Q_{diag})$ is contained in L'_{qs} . We can pass to a splitting covering. By Exp. XXVI Prop. 6.1 H_{S_τ} is smooth with connected fibers; clearly it contains $(T_{qs}^{ad})_{S_\tau}$. By Exp. XXII 5.4.1 such subgroup is uniquely determined by the set of roots α such that the generator e_α of $\text{Lie}((G_{qs})_{S_\tau})$ is contained in its Lie algebra. Note that the restriction of a simple root α_i to Q_{diag} is identity when $i \in J$ and is trivial otherwise. So e_α belongs to $\text{Lie}(H_{S_\tau})$ if and only if the sum of its coefficients at α_i with $i \in J$ is zero. But $(L'_{qs})_{S_\tau}$ is also smooth with connected fibers and corresponds to the same set of roots, hence $L'_{qs} = H$. \square

Proposition 2. *In the setting of Theorem 1, assume moreover that G is simply connected and $\text{Pic}(\text{Dyn}(G)) = 0$. Then $[\xi]$ comes from an element in $H^1(S, G_{qs})$ if and only if $\beta_{G_O}(\omega_O) = 0$ for each orbit O .*

Proof. If $[\xi]$ belongs to the image of $H^1(S, G_{qs}) \rightarrow H^1(S, G_{qs}^{ad})$ then $\delta([\xi]_O) = 0$ and therefore $\beta_{G_O} = 0$ for each O . Conversely, assume that $\beta_{G_O}(\omega_O) = 0$ for each O . Proposition 1 applied to the Borel subgroup implies that $T_{qs} \simeq \prod_O R_{O/S}(\mathbf{G}_m)$ and $T_{qs}^{ad} \simeq \prod_O R_{O/S}(\mathbf{G}_m)$. Now the Shapiro lemma (cf. Exp. XXIV Prop. 8.2) implies that the image of $\delta([\xi])$ in $H^2(S, T_{qs})$ is trivial, while $H^1(S, T_{qs}^{ad}) = \text{Pic}(\text{Dyn}(G)) = 0$. Now the claim follows from the exact sequence

$$H^1(S, T_{qs}^{ad}) \longrightarrow H^2(S, \text{Cent}(G_{qs})) \longrightarrow H^2(S, T_{qs}),$$

which comes from the sequence

$$1 \longrightarrow \text{Cent}(G_{qs}) \longrightarrow T_{qs} \longrightarrow T_{qs}^{ad} \longrightarrow 1.$$

□

- Theorem 2.** (1) *Let (G, u) be an oriented semisimple group scheme of constant type over S , P be its parabolic subgroup admitting a Levi subgroup L , H be the derived subgroup of L with the induced orientation. Denote by Λ the lattice of weights of G_{qs} . For every $\lambda \in \Lambda^*$ denote by λ' the restriction of λ to $\text{Cent}(H_{qs})$. Then $\beta_G(\lambda) = \beta_H(\lambda')$. In particular, for any $\alpha \in \Lambda_r^*$ one has $\beta_H(\alpha') = 0$.*
- (2) *Let G_{qs} be a quasi-split simply connected group, P_{qs} be a standard parabolic subgroup of G_{qs} , L_{qs} be its standard Levi subgroup, H_{qs} be the derived subgroup of L_{qs} . Assume that (H, v) is an oriented inner form of H_{qs} , satisfying $\beta_{H_O}(\alpha'_O) = 0$ for all $O \notin \mathfrak{t}(P_{qs})$. Then there exist an oriented inner form (G, v) of G_{qs} and its parabolic subgroup P admitting a Levi subgroup L such that the derived subgroup of L with the induced orientation is isomorphic to H .*
- (3) *In the setting of (2), assume that $\text{Pic}(\text{Dyn}(S)) = 0$. Then (G, u) is unique up to an isomorphism.*
- (4) *In the setting of (2), assume that S is semilocal. Then (G, u) determines (H, v) up to an isomorphism.*

Proof. 1. Let ξ be a cocycle in $Z^1(S, G_{qs}^{ad})$ corresponding to G , given by elements $g_{\sigma\tau} \in G_{qs}^{ad}(S_\sigma \times_S S_\tau)$ for some covering $\coprod S_\tau \rightarrow S$ that quasi-splits G . Over each S_τ one can (possibly, passing to a finer covering) conjugate P_{S_τ} and L_{S_τ} by some element of G_{qs}^{ad} to P_{qs} and L_{qs} , where P_{qs} is a standard parabolic subgroup of G_{qs} and L_{qs} is its standard Levi subgroup. Adjusting ξ by the coboundary given by these elements, we can assume that all $g_{\sigma\tau}$'s belong to L'_{qs} , where L'_{qs} is the image of L_{qs} in G_{qs}^{ad} , by Exp. XXVI Prop. 1.15 and Cor. 1.8 (cf. Exp. XXVI 3.21)

Let $\rho: G_{qs} \rightarrow \text{GL}(V)$ be a center preserving representation with a weight λ . Consider its restriction to H_{qs} and denote by U the center preserving direct summand corresponding to λ' and by U' its complement invariant under H_{qs} (see Lemma 1, (4)). Denote by T_{qs} the standard maximal torus of L_{qs} and by T'_{qs} its intersection with H_{qs} . Note that U and U' , being sums of weight subspaces of T'_{qs} , are stable under T_{qs} and, therefore, are invariant under the action of L_{qs} . Hence the map $H^1(S, L'_{qs}) \rightarrow H^1(S, \text{PGL}(V))$ factors through $H^1(S, (\text{GL}(U) \times \text{GL}(U'))/\mathbf{G}_m)$, where \mathbf{G}_m is embedded into $\text{GL}(U) \times \text{GL}(U')$ diagonally.

Now the claim is obtained by comparing the diagrams

$$\begin{array}{ccc} H^1(S, (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\mathbf{G}_m) & \longrightarrow & H^2(S, \mathbf{G}_m) \\ \downarrow & & \parallel \\ H^1(S, \mathrm{PGL}(V)) & \longrightarrow & H^2(S, \mathbf{G}_m) \end{array}$$

and

$$\begin{array}{ccc} H^1(S, (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\mathbf{G}_m) & \longrightarrow & H^2(S, \mathbf{G}_m) \\ \downarrow & & \parallel \\ H^1(S, \mathrm{PGL}(U)) & \longrightarrow & H^2(S, \mathbf{G}_m), \end{array}$$

which come from the sequences

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathrm{GL}(U) \times \mathrm{GL}(U') & \longrightarrow & (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\mathbf{G}_m \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathrm{GL}(V) & \longrightarrow & \mathrm{PGL}(V) \longrightarrow 1. \end{array}$$

and

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathrm{GL}(U) \times \mathrm{GL}(U') & \longrightarrow & (\mathrm{GL}(U) \times \mathrm{GL}(U'))/\mathbf{G}_m \longrightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathbf{G}_m & \longrightarrow & \mathrm{GL}(U) & \longrightarrow & \mathrm{PGL}(U) \longrightarrow 1. \end{array}$$

2. Let $[\zeta]$ be the class in $H^1(S, H_{qs}^{ad}) = H^1(S, L_{qs}^{ad})$ corresponding to H . Denote by L'_{qs} and H'_{qs} the images of L_{qs} and H_{qs} in G_{qs}^{ad} . Let us compute the image $\delta([\zeta]) \in H^2(S, \mathrm{Cent}(L'_{qs}))$. Using the assumption, Theorem 1 (2), and the commutative diagram

$$\begin{array}{ccc} H^1(S, H_{qs}^{ad}) & \xrightarrow{\delta} & H^2(S, \mathrm{Cent}(H'_{qs})) \\ \parallel & & \downarrow \\ H^1(S, L_{qs}^{ad}) & \xrightarrow{\delta} & H^2(S, \mathrm{Cent}(L'_{qs})), \end{array}$$

we see that $(\alpha_O)_* \delta([\zeta_O]) = 0$ for any $O \notin \mathfrak{t}(P_{qs})$. Now Proposition 1 (2) and the Shapiro lemma show that $\delta([\zeta]) = 0$. It means that $[\zeta]$ comes from some $[\xi] \in H^1(S, L'_{qs})$, and the image of $[\xi]$ in $H^1(S, G_{qs}^{ad})$ defines the desired G .

3. Let (G, u) be such a group; denote by $[\xi]$ the corresponding class in $H^1(S, G_{qs}^{ad})$. As we have seen in the proof of (1), $[\xi]$ comes from an element of $H^1(S, L'_{qs})$, say $[\zeta]$. We have to show that $[\zeta]$ (and a fortiori $[\xi]$) is completely determined by its image in $H^1(S, L_{qs}^{ad})$, or, in other words, that the canonical map $\pi_*: H^1(S, L'_{qs}) \rightarrow H^1(S, L_{qs}^{ad})$ is injective. Since $\mathrm{Cent}(L'_{qs})$ is central in L'_{qs} , $H^1(S, \mathrm{Cent}(L'_{qs}))$ acts on $H^1(S, L'_{qs})$, and the orbits of the action coincide with the fibers of π_* . But $H^1(S, \mathrm{Cent}(L'_{qs}))$ by Proposition 1 (2) and the Shapiro lemma injects into $\mathrm{Pic}(\mathrm{Dyn}(G))$, which is trivial by the assumption.

4. Follows from the proof of (3) and the fact that the map $H^1(S, L'_{qs}) \rightarrow H^1(S, G_{qs}^{ad})$ is injective (Exp. XXVI Cor. 5.10). \square

5. COMBINATORIAL RESTRICTIONS

From now on we assume that $S = \operatorname{Spec} R$, where R is a connected semilocal ring. Recall that in this case all minimal parabolic subgroups P_{\min} of G are conjugate under $G(S)$ and hence have the same type $\mathfrak{t}_{\min} = \mathfrak{t}(P_{\min})$, which is a clopen subscheme of $\operatorname{Dyn}(G)$ (Exp. XXVI Cor. 5.7). By Exp. XXVI Lemme 3.8 $P \mapsto \mathfrak{t}(P)$ is a bijection between parabolic subgroups P of G containing P_{\min} and clopen subschemes \mathfrak{t} of $\operatorname{Dyn}(G)$ containing \mathfrak{t}_{\min} .

Since S is affine, for any parabolic subgroup P of G there exists a Levi subgroup L (Exp. XXVI Cor. 2.3) of P , and a unique parabolic subgroup P^- which is opposite to P with respect to L , i.e. satisfies $P^- \cap P = L$ (Exp. XXVI Th. 4.3.2). The type $\mathfrak{t}(P^-)$ is the image $s_G(\mathfrak{t}(P))$ of $\mathfrak{t}(P)$ under an automorphism s_G of $\operatorname{Dyn}(G)$ called the *opposition involution* (Exp. XXIV Prop. 3.16.6 and Exp. XXVI 4.3.1; cf. [11] 1.5.1). The corresponding automorphism $s_G \in \operatorname{Aut}(D)$ is induced by the automorphism $\alpha \mapsto -w_0(\alpha)$ of the root system Φ of G_0 , where w_0 is the unique element of maximal length in the Weyl group of Φ . In fact s_G acts nontrivially only on irreducible components of Φ of type A_n , $n \geq 2$, D_{2n+1} , $n \geq 1$, or E_6 , where it coincides with the unique nontrivial automorphism of the component.

By the *Tits index* of G we mean the pair $(\operatorname{Dyn}(G), \mathfrak{t}_{\min})$. Clearly, we have $\mathfrak{t}_{\min} = s_G(\mathfrak{t}_{\min})$, since if $P = P_{\min}$ is a minimal parabolic subgroup, then P^- is also minimal.

The group G is quasi-split if \mathfrak{t}_{\min} is empty. In the opposite case when $\mathfrak{t}_{\min} = \operatorname{Dyn}(G)$ we say that G is *anisotropic*. The *anisotropic kernel* G_{an} of G is defined as the derived subgroup of a Levi part of P_{\min} , which is indeed anisotropic by Exp. XXVI Prop. 1.20.

Tits indices can be described in the set-theoretic style as follows. The assumption that $S = \operatorname{Spec} R$ is connected allows us to identify D_T with D , and a clopen $*$ -invariant subscheme of D_T with a $*$ -invariant subset of D . Let $J \subseteq D$ be the complement of the subset corresponding to $(\mathfrak{t}_{\min})_T$. Then the Tits index of G is determined by the pair (D, J) together with a $*$ -action on D , represented by a subgroup Γ of $\operatorname{Aut}(D)$. Usually we indicate Γ by writing its order as the upper left index attached to D (for example, 2E_6 , 6D_4 and so on). The group G is *of inner type* if $\operatorname{Dyn}(G) \simeq \operatorname{Dyn}(G_0)$, or, in other words, $\Gamma = \{1\}$.

From now on, we fix a minimal parabolic subgroup $P = P_{\min}$ of G , a Levi subgroup L of P , and a maximal split subtorus Q of G such that $L = \operatorname{Cent}_G(Q)$, which exists by Exp. XXVI Cor. 6.11 (or by Proposition 1 (3) and descent). Let M be the lattice of characters of Q . The Lie algebra $\operatorname{Lie}(G)$ of G decomposes under the action of Q into a direct sum of weight subspaces:

$$\operatorname{Lie}(G) = \operatorname{Lie}(L) \oplus \bigoplus_{\alpha \in M \setminus \{0\}} \operatorname{Lie}(G)^\alpha.$$

We denote by Ψ the set of elements $\alpha \in M \setminus \{0\}$ such that $\operatorname{Lie}(G)^\alpha \neq 0$. By Exp. XXVI Th. 7.4 Ψ is a root system, which is called the *relative root system* of G with respect to Q . One readily sees that the simple roots of Ψ correspond bijectively to the $*$ -orbits contained in J .

Denote by \hat{D} the extended Dynkin diagram (one adds a vertex corresponding to minus the maximal root to each irreducible component of D), and by \hat{J} the union $J \cup (\hat{D} \setminus D)$.

Lemma 3. *Let G be a semisimple algebraic group over S , and let (D, J) be the Tits index of G . Then any $*$ -orbit $O \subseteq \hat{J}$ is invariant under the opposition involution of the Dynkin diagram $(\hat{D} \setminus \hat{J}) \cup O$.*

Proof. Let $A \in \Psi$ be the relative root corresponding to O (it is simple if $O \subseteq J$ and the opposite to the maximal otherwise). By Exp. XXVI Prop. 6.1 the subsets $\mathbb{Z}A \cap \Psi$ and $\mathbb{Z}A \cap \Psi^+$ correspond to certain subgroups G' and P' of G ; moreover, G' is reductive and P' is a parabolic subgroup of G' having L as a Levi subgroup. Since L is anisotropic, P' is a minimal parabolic subgroup of G' . Passing to a splitting covering one sees that the Dynkin diagram of G' is $(\hat{D} \setminus \hat{J}) \cup O$, and the type of P' is given by O . The Lemma follows. \square

In the next Proposition we list all possible cases when the conclusion of Lemma 3 holds for an irreducible root system Φ . Our numbering of the vertices of Dynkin diagrams follows [1].

Proposition 3. *Let Φ be a reduced irreducible root system, D the corresponding Dynkin diagram, $J \neq \emptyset$ a subset of D and Γ a group of automorphisms of D . A triple (Φ, J, Γ) satisfies that any Γ -orbit $O \subseteq \hat{J}$ is invariant under the opposition involution of the Dynkin diagram $(\hat{D} \setminus \hat{J}) \cup O$, if and only if it is, up to an automorphism of D , one in the following list:*

- (1) $\Phi = A_n$, $n \geq 1$; $|\Gamma| = 1$; $J = \{d, 2d, \dots, rd\}$ for some $d, r \geq 1$ such that $d \cdot (r+1) = n+1$.
- (2) $\Phi = A_n$, $n \geq 2$; $|\Gamma| = 2$; $J = \{d, 2d, \dots, rd, n+1-d, n+1-2d, \dots, n+1-rd\}$ for some $d, r \geq 1$ such that $d \mid n+1$, $2rd \leq n+1$.
- (3) $\Phi = B_n$, $n \geq 2$; $|\Gamma| = 1$; $J = \{d, 2d, \dots, rd\}$ for some $d, r \geq 1$ such that d is even or $d = 1$, $rd \leq n$.
- (4) $\Phi = C_n$, $n \geq 2$; $|\Gamma| = 2$; $J = \{d, 2d, \dots, rd\}$ for some $d, r \geq 1$ such that $rd \leq n$.
- (5) $\Phi = D_n$, $n \geq 4$; $|\Gamma| = 1$; $J = \{d, 2d, \dots, rd\}$ for some $d, r \geq 1$ such that d is even or $d = 1$, $rd \leq n$, $rd \neq n-1$.
- (6) $\Phi = D_n$, $n \geq 4$; $|\Gamma| = 2$; $J = \{d, 2d, \dots, rd\}$ (or $J = \{d, 2d, \dots, (r-2)d, n-1, n\}$ in the case $rd = n-1$) for some $d, r \geq 1$ such that d is even or $d = 1$, $rd \leq n-1$.
- (7) $\Phi = D_4$; $|\Gamma| = 3$ or $|\Gamma| = 6$; $J = \{2\}$, D .
- (8) $\Phi = E_6$; $|\Gamma| = 1$; $J = \{2\}, \{1, 6\}, \{2, 4\}$, D .
- (9) $\Phi = E_6$; $|\Gamma| = 2$; $J = \{2\}, \{1, 6\}, \{2, 4\}, \{1, 6, 2\}$, D .
- (10) $\Phi = E_7$; $|\Gamma| = 1$; $J = \{1\}, \{6\}, \{7\}, \{1, 3\}, \{1, 6\}, \{1, 6, 7\}, \{1, 3, 4, 6\}$, D .
- (11) $\Phi = E_8$; $|\Gamma| = 1$; $J = \{1\}, \{8\}, \{1, 8\}, \{7, 8\}, \{1, 6, 7, 8\}$, D .
- (12) $\Phi = F_4$; $|\Gamma| = 1$; $J = \{1\}, \{4\}, \{1, 4\}$, D .
- (13) $\Phi = G_2$; $|\Gamma| = 1$; $J = \{2\}$, D .

Proof. If Φ is an exceptional root system or D_4 , the result is verified by an easy try-out. Consider the case $\Phi = A_n$, $|\Gamma| = 1$. The opposition involution of A_n is the non-trivial automorphism of D , hence if $|J| = 1$ then $n = 2k+1$ and $J = \{k+1\}$, the middle vertex. Proceeding by induction on $|J|$, we see that $J = \{d, 2d, \dots, rd\}$ for some $d \geq 1$ such that $d \mid n+1$, $d \cdot (r+1) = n+1$, and any such J is valid. If $|\Gamma| = 2$ then since J is Γ -invariant, J contains a vertex k if and only if it contains

$n + 1 - k$; the opposition involution condition implies that $J = \{d, 2d, \dots, rd\} \cup \{n + 1 - d, n + 1 - 2d, \dots, n + 1 - rd\}$, and any such J is valid.

Now consider the case $\Phi = B_n, C_n, D_n$ and $|\Gamma| = 1$. Let $J = \{i_1, i_2, \dots, i_r\}$, $i_1 < i_2 < \dots < i_r$. If $\Phi = D_n$ and $i_r > n - 2$, we may assume $i_r = n$ applying an automorphism of D . Then $J \setminus \{i_r\}$ lies in the connected component of $D \setminus \{i_r\}$ of type A_{i_r-1} . Since $J \setminus \{i_r\}$ satisfies the opposition involution condition, by the A_n case $J \setminus \{i_r\}$ is of the form $\{d, 2d, \dots, (r-1)d\}$ for some $d \geq 1$ such that $i_r = rd$. Therefore, $J = \{d, 2d, \dots, rd\}$, as required. If $\Phi = C_n$, this finishes the proof, since any such J satisfies the opposition involution condition. If $\Phi = D_n$ or B_n , such J does not satisfy the opposition involution condition for $O = \hat{J} \setminus J$ if d is odd > 1 , so this case is excluded. The case $\Phi = D_n$, $|\Gamma| = 2$ is verified analogously. \square

6. TITS INDICES

We now start the classification of semisimple algebraic groups over $S = \text{Spec } R$, where R is a connected semilocal ring. The problem allows two immediate reductions. First, every semisimple group G is completely determined by its root datum and the corresponding simply connected group G^{sc} , so we can assume that G is simply connected.

Second, if the Dynkin diagram D of G is not connected (that is, the root system is not irreducible), we can present D as the disjoint union of its *isotypic* components D_t (it means that we collect isomorphic components together), and then we have a canonical decomposition $G \simeq \prod G_t$, where G_t is a group over S with the Dynkin diagram D_t (Exp. XXIV Prop. 5.5). Further, if D_t is the disjoint union of n_t copies of a connected graph $D_{0,t}$, there exists a canonical étale extension S_t/S of degree n_t and a group $G_{0,t}$ over S_t such that $G_t \simeq R_{S_t/S}(G_{0,t})$ (Exp. XXIV Prop. 5.9). So we can assume that D is connected, that is, G is a *simple* algebraic group.

Our reasoning will be based on Theorem 2, which implies that an oriented semisimple simply connected algebraic group G is determined, up to an isomorphism, by its Tits index and the isomorphism class of its anisotropic kernel G_{an} , subject to certain conditions on the Tits algebras, together with an isomorphism $\text{Dyn}(G_{an}) \simeq \mathfrak{t}_{\min}$. Thus the classification consists in listing all possible Tits indices of simple algebraic groups, and, for any given index, the conditions on the corresponding anisotropic kernels. The necessary combinatorial restriction on a Tits index stated in Lemma 3 reduces possibilities to those listed in Proposition 3. For some of them conditions on the Tits algebras lead to a contradiction; for the rest they give criteria that anisotropic kernels must satisfy.

We represent Tits indices graphically by Dynkin diagrams D with the vertices in J being circled; nontrivial $*$ -action is indicated by arrows \longleftrightarrow . We also use the Tits notation ${}^m X_{n,r}^k$ for the groups of specific indices (see [11]).

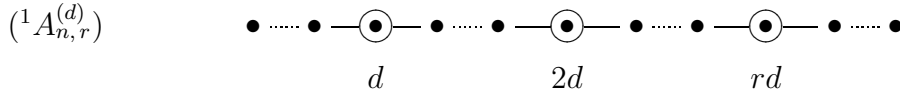
We begin with simple groups of type A_n . The split simple simply connected group of type A_n over R is $\text{SL}_{n+1}(R)$; the corresponding adjoint group is $\text{PGL}_{n+1}(R) = \text{Aut}(\text{M}_{n+1}(R))$. So the oriented simple simply connected groups of inner type A_n are of the form $\text{SL}_1(A)$, where A is an Azumaya algebra over R of degree $n + 1$, uniquely determined up to an isomorphism. Obviously A is the Tits algebra of $\text{SL}_1(A)$ corresponding to the natural representation of $\text{SL}_{n+1}(R)$ in R^{n+1} ; so $[A] =$

$\beta_{\mathrm{SL}_1(A)}(\omega_1)$. The change of orientation corresponds to the replacement of A with A^{op} . Note that $\mathrm{SL}_1(A) \simeq \mathrm{SL}_1(A^{\mathrm{op}})$ as groups, the isomorphism being $g \mapsto g^{-1}$.

Lemma 4. *Assume that $\mathrm{SL}_1(E)$ and $\mathrm{SL}_1(E')$ are anisotropic, and $[E] = [E']$ in $\mathrm{Br}(R)$. Then $E \simeq E'$.*

Proof. Since projective modules over R are free, $[E] = [E']$ means that $M_n(E) \simeq M_m(E')$ for some n and m . Then $\mathrm{SL}_n(E)$ and $\mathrm{SL}_m(E')$ are isomorphic as oriented groups. Now $\mathrm{SL}_1(E)^n$ and $\mathrm{SL}_1(E')^m$ are both anisotropic kernels of G , so they are isomorphic. In particular, they have the same type, that is $m = n$, and the degrees of E and E' are equal. Theorem 2 (4) implies that $\mathrm{SL}_1(E)^m$ and $\mathrm{SL}_1(E')^m$ are isomorphic as oriented groups, hence $\mathrm{SL}_1(E)$ and $\mathrm{SL}_1(E')$ are isomorphic as oriented groups, that is $E \simeq E'$. \square

Theorem 3 (${}^1\mathbf{A}_n$). *Every simple simply connected group G of inner type A_n over R has the Tits index $({}^1A_n, J)$, where $J = \{d, 2d, \dots, rd\}$, $n + 1 = (r + 1)d$:*



G is isomorphic to $\mathrm{SL}_{r+1}(E)$, and the anisotropic kernel is $\mathrm{SL}_1(E)^r$, where $\deg E = d$.

Proof. Let $({}^1A_n, J)$ be the Tits index of G ; we have $J = \{d, 2d, \dots, rd\}$ for some d with $n + 1 = (r + 1)d$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \dots \times \mathrm{SL}_1(E_{r+1})$ for some Azumaya algebras E_1, \dots, E_r . The Cartan matrix of A_n shows that $\alpha_{i,d} = 2\omega_{i,d} - \omega_{i,d-1} - \omega_{i,d+1}$ for $i = 1, \dots, r$. By Theorem 2, we have

$$0 = \beta_{G_{\mathrm{an}}}(\alpha'_{i,d}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Now Lemma 4 implies that all E_i are isomorphic. Set $E = E_1$; then $\mathrm{SL}_{r+1}(E)$ has the same Tits index and the same anisotropic kernel as G , so by Theorem 2 we have $G \simeq \mathrm{SL}_{r+1}(E)$, as claimed. \square

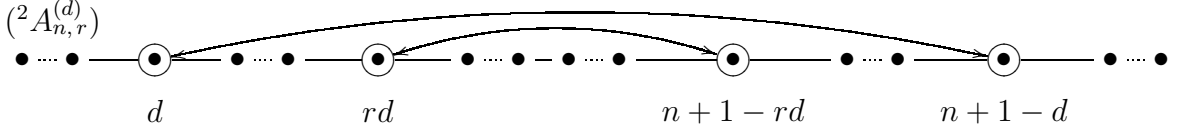
The above result implies that for any Azumaya algebra A over R , the group $G = \mathrm{SL}_1(A)$ is isomorphic to $\mathrm{SL}_{r+1}(E)$, where E is an Azumaya algebra such that $\mathrm{SL}_1(E)$ is anisotropic. In this case the degree of E is called the *index* of A and is denoted by $\mathrm{ind} A$; obviously $\mathrm{ind} A$ divides $\deg A$. The *exponent* $\exp A$ of A is the order of $[A]$ in $\mathrm{Br}(R)$. We will need the following result:

Proposition 4. *Let A be an Azumaya algebra. Then $\exp A$ divides $\mathrm{ind} A$, and they have the same prime factors.*

Proof. The first part follows from the fact that $[A] = [E] = \beta_{\mathrm{SL}_1(E)}(\omega_1)$, and $(\deg E)\omega_1$ belongs to the root lattice of $\mathrm{SL}_1(A)$. The second part follows from [3, Ch. II, Thm. 1]. \square

Let R'/R be an étale extension of degree n . We can interpret the corestriction homomorphism $\mathrm{cores}_{R'/R}: \mathrm{Br}(R') \rightarrow \mathrm{Br}(R)$ as follows. If A is an Azumaya algebra over R' of degree d , $R_{R'/R}(\mathrm{SL}_1(A))$ is a group of type nA_{d-1} over R , with the $*$ -action permuting the copies of A_{d-1} . Now $\mathrm{cores}_{R'/R}([A]) = \beta_{R_{R'/R}(\mathrm{SL}_1(A))}(\omega)$, where ω is the sum of the fundamental weights ω_1 of all copies of A_{d-1} (cf. [12, § 5.3]).

Theorem 3 (${}^2\mathbf{A}_n$). *Every simple simply connected group G of type 2A_n over R has the Tits index $({}^2A_n, J)$, where $J = \{d, 2d, \dots, rd, n+1-rd, \dots, n+1-2d, n+1-d\}$ for some $r \geq 0, d > 0$ such that $d \mid n+1, 2rd \leq n+1$:*



Denote by $\text{Spec } R'$ the orbit corresponding to $\{1, n\}$ (so that R'/R is a connected quadratic étale extension). The possible anisotropic kernels are the following:

- $H \times R_{R'/R}(\text{SL}_1(E))^r$, where E is an Azumaya algebra over R' with $\text{ind } E = \text{deg } E = d$, H is a simple simply connected anisotropic of type ${}^2A_{n-2rd}$ over R whose orbit O corresponding to $\{1, n-2rd\}$ is isomorphic to $\text{Spec } R'$, such that $\beta_{H_O}(\omega_1) = [E]$, when $n-2rd \geq 2$;
- $\text{SL}_1(A) \times R_{R'/R}(\text{SL}_1(A_{R'}))^r$, where A is an Azumaya algebra over R such that $\text{ind } A = \text{deg } A = 2$ and $\text{ind } A_{R'} = d$, when $n-2rd = 1$;
- $R_{R'/R}(\text{SL}_1(E))^r$, where E is an Azumaya algebra over R such that $\text{ind } E = \text{deg } E = d$ and $\text{cores}_{R'/R}([E]) = 0$, when $n-2rd \leq 0$.

Proof. Let $({}^2A_n, J)$ be the Tits index of G ; we have $J = \{d, 2d, \dots, rd, n+1-rd, \dots, n+1-2d, n+1-d\}$ for some $r \geq 0, d > 0$ with $d \mid n+1, 2rd \leq n+1$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $H_1 \times \dots \times H_r \times H$, where H_i are groups of outer type $A_{d-1} + A_{d-1}$ with the $*$ -action permuting two summands, and H is a group of outer type ${}^2A_{n-2rd}$ when $n-2rd \geq 2$, is isomorphic to $\text{SL}_1(A)$ for some Azumaya algebra A over R with $\text{ind } A = \text{deg } A = 2$ when $n-2rd = 1$, and is trivial otherwise. Over R' every H_i becomes inner, hence we have $H_i \simeq R_{R'/R}(\text{SL}_1(E_i))$ for some Azumaya algebra E_i over R' , $\text{ind } E_i = \text{deg } E_i = d$.

Denote the orbit corresponding to $\{i \cdot d, n+1-i \cdot d\}$ by $O_i, i = 1, \dots, r$. The Cartan matrix of A_n shows that $\alpha_{i \cdot d} = 2\omega_{i \cdot d} - \omega_{i \cdot d-1} - \omega_{i \cdot d+1}$. When $i < r$, by Theorem 2 we have

$$0 = \beta_{(G_{an})_{O_i}}(\alpha'_{O_i}) = \beta_{\text{SL}_1(E_i)}(\omega_1) - \beta_{\text{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; we set $E = E_1$.

In the case $n-2rd \geq 2$ by Theorem 2 we have

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\text{SL}_1(E)}(\omega_1) - \beta_{H_{O_r}}(\omega_1) = [E] - \beta_{H_{O_r}}(\omega_1).$$

In the case $n-2rd = 1$ we have

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\text{SL}_1(E)}(\omega_1) - \beta_{\text{SL}_1(A)_{O_r}}(\omega_1) = [E] - [A_{R'}],$$

for $O_r \simeq \text{Spec } R'$ as a scheme.

In the case $n-2rd = 0$ we have

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\text{SL}_1(E)}(\omega_1) = [E],$$

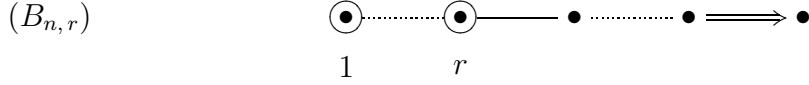
hence $E \simeq R'$. G is quasi-split in this case.

Finally, in the case $n-2rd = -1$ we have $O_r \simeq \text{Spec } R$, and hence

$$0 = \beta_{G_{an}}(\alpha'_{O_r}) = \text{cores}_{R'/R}(\beta_{\text{SL}_1(E)}(\omega_1)) = \text{cores}_{R'/R}([E]).$$

□

Theorem 3 (B_n). *Every simple simply connected group of type B_n over R , $n \geq 2$, has the Tits index (B_n, J) , where $J = \{1, 2, \dots, r\}$ for some $r \geq 0$:*



The possible anisotropic kernels are the following:

- simple simply connected anisotropic groups of type B_{n-r} over R , when $n-r \geq 2$;
- $SL_1(A)$, where A is an Azumaya algebras A over R with $\text{ind } A = \deg A = 2$, when $n-r = 1$.

If $n = r$ then G is split.

Proof. Let (B_n, J) be the Tits index of G ; we have $J = \{d, 2d, \dots, rd\}$ for some $r \geq 0$, $d > 0$ with $rd \leq n$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $SL_1(E_1) \times \dots \times SL_1(E_r) \times H$, where H is a group of type B_{n-rd} when $n-rd \geq 2$, is isomorphic to $SL_1(A)$ for some Azumaya algebra A over R with $\text{ind } A = \deg A = 2$ when $n-rd = 1$, or is trivial when $n = rd$.

In the case $n-rd \geq 2$ the Cartan matrix of B_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$. By Theorem 2, we have

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{SL_1(E_r)}(\omega_1) - \beta_H(\omega_1) = [E_r].$$

So $E_r = R$, hence $d = 1$.

In the case $n-rd = 1$ we have $\alpha_{rd} = 2\omega_{n-1} - \omega_{n-2} - 2\omega_n$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{SL_1(E_r)}(\omega_1) - 2\beta_H(\omega_1) = [E_r],$$

and again $d = 1$.

Finally, in the case $n = rd$ we have $\alpha_{rd} = 2\omega_n - \omega_{n-1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{SL_1(E_r)}(\omega_1) = [E_r],$$

$d = 1$, and G is split in this case. □

The split simple simply connected group scheme of type C_n over R is $Sp_{2n}(R)$.

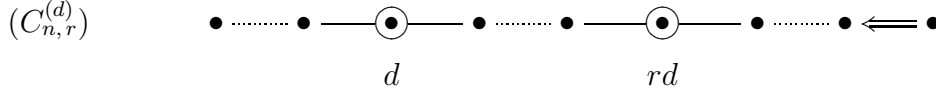
Proposition 5. *Assume that G is a simple simply connected group of type C_n over R , $\beta_G(\omega_1) = [E]$, $\text{ind } E = d$. Then $d = 2^k$ for some $k \geq 0$ and $d \mid 2n$. If $d = 1$ then G is split.*

Proof. We have $2[E] = 0$, since $2\omega_1$ belongs to Λ_r . Now Proposition 4 implies that $d = 2^k$.

The vector representation $\rho: Sp_{2n}(R) \rightarrow GL(R^{2n})$ is center preserving and has a weight ω_1 ; so $[A_\rho] = [E]$. But A_ρ has degree $2n$, so $d \mid 2n$.

If $d = 1$ then by Proposition 2 G corresponds to an element of $H^1(R, Sp_{2n})$, and the latter is trivial by [7, Ch. I, Cor. 4.1.2]. □

Theorem 3 (C_n). *Every simple simply connected group G of type C_n over R , $n \geq 2$, has the Tits index (C_n, J) , where $J = \{d, 2d, \dots, rd\}$ for some $r \geq 0$, $d > 0$ such that $d = 2^k \mid 2n$, $rd \leq n$, and $r = n$ when $d = 1$:*



The possible anisotropic kernels are the following:

- $H \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, H is a simple simply connected anisotropic of type C_{n-rd} over R with $\beta_H(\omega_1) = [E]$, when $n - rd \geq 2$;
- $\mathrm{SL}_1(E)^{r+1}$, where E is an Azumaya algebras E over R with $\mathrm{ind} E = \deg E = d$, when $n - rd = 1$;
- $\mathrm{SL}_1(E)^r$, where E is an Azumaya algebras E over R with $\mathrm{ind} E = \deg E = d$ and $\exp E \leq 2$, when $n - rd = 0$.

Proof. Let (C_n, J) be the Tits index of G ; we have $J = \{d, 2d, \dots, rd\}$ for some $r \geq 0$, $d > 0$ with $rd \leq n$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \dots \times \mathrm{SL}_1(E_r) \times H$, where H is a group of type C_{n-rd} when $n - rd \geq 2$, is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\mathrm{ind} A = \deg A = 2$ when $n - rd = 1$, or is trivial when $n = rd$.

The Cartan matrix of C_n shows that $\alpha_{i,d} = 2\omega_{i,d} - \omega_{i,d-1} - \omega_{i,d+1}$ for $i = 1, \dots, r-1$. By Theorem 2, we have

$$0 = \beta_{G_{an}}(\alpha'_{i,d}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; set $E = E_1$. Note that $[E] = \beta_G(\omega_1)$, hence by Proposition 5 $d = 2^k \mid 2n$, and G is split when $d = 1$.

In the case $n - rd \geq 2$ the Cartan matrix of C_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

In the case $n - rd = 1$ we have $\alpha_{rd} = 2\omega_{n-1} - \omega_{n-2} - \omega_n$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)}(\omega_1) = [E] - [A].$$

Hence $[E] = [A]$ and $d = 2$.

Finally, in the case $n = rd$ we have $\alpha_{rd} = 2\omega_n - 2\omega_{n-1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = 2\beta_{\mathrm{SL}_1(E)}(\omega_1) = 2[E],$$

that is $\exp E \leq 2$. □

The split simple simply connected group scheme of type D_n over R is $\mathrm{Spin}_{2n}(R)$.

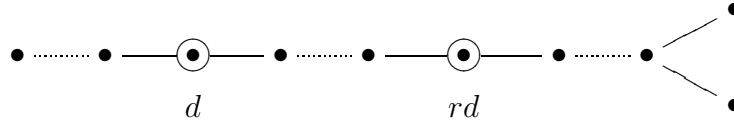
Proposition 6. *Assume that G is a simple simply connected group of type 1D_n or 2D_n over R , $n \geq 4$, $\beta_G(\omega_1) = [E]$, $\mathrm{ind} E = d$. Then $d = 2^k$ for some $k \geq 0$ and $d \mid 2n$.*

Proof. We have $2[E] = 0$, since $2\omega_1$ belongs to Λ_r . Now Proposition 4 implies that $d = 2^k$.

The vector representation $\rho: \mathrm{Spin}_{2n}(R) \rightarrow \mathrm{GL}(R^{2n})$ is center preserving and has a weight ω_1 ; so $[A_\rho] = [E]$. But A_ρ has degree $2n$, so $d \mid 2n$. □

Theorem 3 (1D_n). *Every simple simply connected group G of inner type D_n over R , $n \geq 4$, has the Tits index $({}^1D_n, J)$, where $J = \{d, 2d, \dots, rd\}$ (possibly, after interchanging $n-1$ and n) for some $r \geq 0$, $d > 0$ such that $d = 2^k \mid 2n$, $rd \leq n$, $n \neq rd + 1$:*

$({}^1D_{n,r}^{(d)})$



The possible anisotropic kernels are the following:

- $H \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, H is a simple simply connected anisotropic group of inner type D_{n-rd} over R with $\beta_H(\omega_1) = [E]$, when $n - rd \geq 4$;
- $\mathrm{SL}_1(A) \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, A is an Azumaya algebra over R with $\mathrm{ind} A = \deg A = 4$ such that $2[A] = [E]$, when $n - rd = 3$;
- $\mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2) \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, A_1 and A_2 are Azumaya algebras over R such that $\mathrm{ind} A_1 = \deg A_1 = \mathrm{ind} A_2 = \deg A_2 = 2$ and $[A_1] + [A_2] = [E]$, when $n - rd = 2$;
- $\mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$ and $\exp E \leq 2$, when $n = rd$.

Proof. Let $({}^1D_n, J)$ be the Tits index of G ; we have $J = \{d, 2d, \dots, rd\}$ for some $r \geq 0$, $d > 0$ with $rd \leq n$, $rd \neq n - 1$ by Lemma 3 and Proposition 3. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(E_1) \times \dots \times \mathrm{SL}_1(E_r) \times H$, where H is a group of inner type D_{n-rd} when $n - rd \geq 4$, is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\mathrm{ind} A = \deg A = 4$ when $n - rd = 3$ is isomorphic to $\mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2)$ for some Azumaya algebras A_1, A_2 over R with $\mathrm{ind} A_1 = \deg A_1 = \mathrm{ind} A_2 = \deg A_2 = 2$, or is trivial when $n = rd$.

The Cartan matrix of D_n shows that $\alpha_{i,d} = 2\omega_{i,d} - \omega_{i,d-1} - \omega_{i,d+1}$ for $i = 1, \dots, r-1$. By Theorem 2, we have

$$0 = \beta_{G_{an}}(\alpha'_{i,d}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

Lemma 4 implies now that all E_i are isomorphic; set $E = E_1$. Note that $[E] = \beta_G(\omega_1)$, hence by Proposition 6 $d = 2^k \mid 2n$.

In the case $n - rd \geq 4$ the Cartan matrix of D_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

In the case $n - rd = 3$ we still have $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A)}(\omega_2) = [E] - 2[A].$$

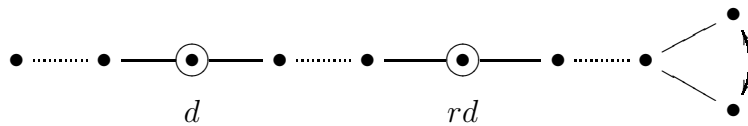
In the case $n - rd = 2$ we have $\alpha_{rd} = 2\omega_{n-2} - \omega_{n-3} - \omega_{n-1} - \omega_n$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_2)}(\omega_2) = [E] - [A_1] - [A_2].$$

Finally, in the case $n = rd$ we have $\alpha_{rd} = 2\omega_n - \omega_{n-2}$, so

$$0 = \beta_{G_{an}}(\alpha'_{rd}) = \beta_{\mathrm{SL}_1(E)}(\omega_2) = 2[E],$$

hence $\exp E \leq 2$. □

$$\binom{2}{D_{n,r}^{(d)}}$$


- $H \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, H is a simple simply connected anisotropic group of type ${}^2D_{n-rd}$ over R whose orbit corresponding to $\{n - rd - 1, n - rd\}$ is isomorphic to $\mathrm{Spec} R'$, such that $\beta_H(\omega_1) = [E]$, when $n - rd \geq 4$;
- $H \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, H is a simple simply connected anisotropic group of type 2A_3 over R whose orbit corresponding to $\{1, 3\}$ is isomorphic to $\mathrm{Spec} R'$, such that $\beta_H(\omega_2) = [E]$, when $n - rd = 3$;
- $R_{R'/R}(\mathrm{SL}_1(A)) \times \mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = d$, A is an Azumaya algebras over R' such that $\mathrm{ind} A = \deg A = 2$ and $\mathrm{cores}_{R'/R}([A]) = [E]$, when $n - rd = 2$;
- $\mathrm{SL}_1(E)^r$, where E is an Azumaya algebra over R such that $\mathrm{ind} E = \deg E = d$ and $[E_{R'}] = 0$, when $n - rd = 1$.

Denote the orbit corresponding to $\{i \cdot d, n + 1 - i \cdot d\}$ by O_i , $i = 1, \dots, r$. The Cartan matrix of D_n shows that $\alpha_{i \cdot d} = 2\omega_{i \cdot d} - \omega_{i \cdot d-1} - \omega_{i \cdot d+1}$ for $i = 1, \dots, r - 1$. By Theorem 2, we have

$$0 = \beta_{(G_{an})_{O_i}}(\alpha'_{O_i}) = \beta_{\mathrm{SL}_1(E_i)}(\omega_1) - \beta_{\mathrm{SL}_1(E_{i+1})}(\omega_1) = [E_i] - [E_{i+1}].$$

In the case $n - rd \geq 4$ the Cartan matrix of D_n shows that $\alpha_{rd} = 2\omega_{rd} - \omega_{rd-1} - \omega_{rd+1}$, so

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_1) = [E] - \beta_H(\omega_1).$$

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\mathrm{SL}_1(E)}(\omega_1) - \beta_H(\omega_2).$$

In the case $n - rd = 2$ $O_r \simeq \text{Spec } R$, and we have $\alpha_{rd} = 2\omega_{n-2} - \omega_{n-3} - \omega_{n-1} - \omega_n$, so

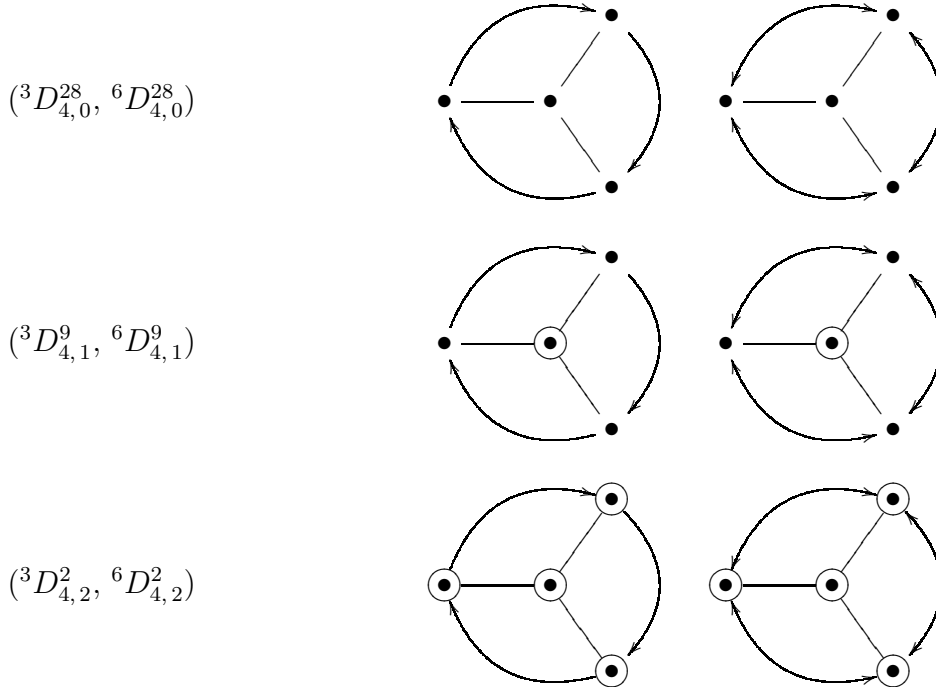
$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\text{SL}_1(E)}(\omega_1) - \text{cores}_{R'/R}(\beta_{\text{SL}_1(A)}(\omega_1)) = [E] - \text{cores}_{R'/R}([A]).$$

Finally, in the case $n - rd = 1$ the condition $d|2n$ implies $d \in \{1, 2\}$; also, $O_r \simeq \text{Spec } R'$, and we have $\alpha_{rd} = 2\omega_n - \omega_{n-2}$, so

$$0 = \beta_{(G_{an})_{O_r}}(\alpha'_{O_r}) = \beta_{\text{SL}_1(E)_{O_r}}(\omega_1) = [E_{R'}].$$

□

Theorem 3 (${}^3\mathbf{D}_4$ and ${}^6\mathbf{D}_4$). *Every simple simply connected group G of type 3D_4 or 6D_4 over R has one of the following Tits indices:*



Denote by $\text{Spec } R'$ the orbit corresponding to $\{1, 3, 4\}$ (so that R'/R is a connected cubic étale extension). The possible anisotropic kernels in the case of ${}^3D_{4,1}^9$ or ${}^6D_{4,1}^9$ are of the form $R_{R'/R}(\text{SL}_1(A))$, where A is an Azumaya algebra over R' with $\text{ind } A = \deg A = 2$ and $\text{cores}_{R'/R}([A]) = 0$.

In the case of ${}^3D_{4,0}^{28}$ or ${}^6D_{4,0}^{28}$ G is anisotropic; in the case of ${}^3D_{4,2}^2$ or ${}^6D_{4,2}^2$ G is quasi-split.

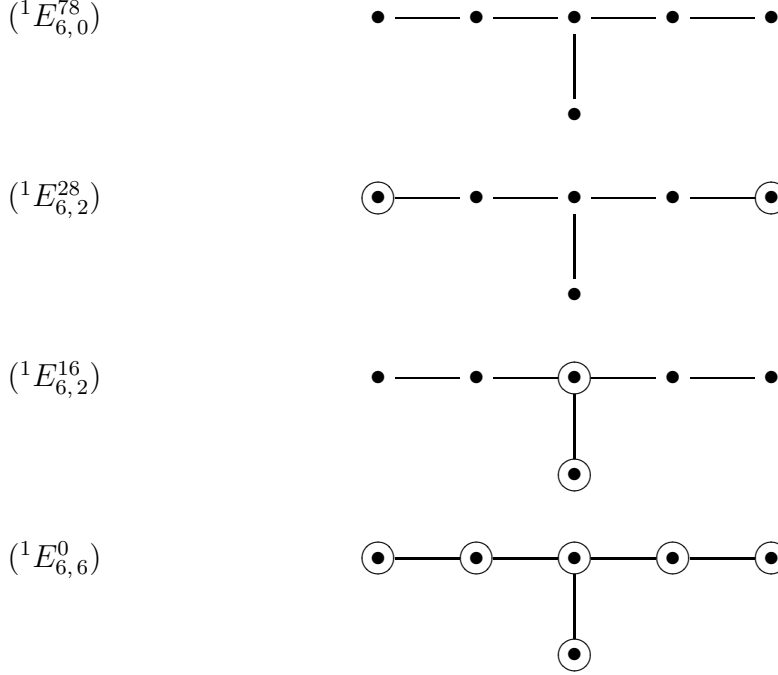
Proof. By Lemma 3 and Proposition 3 G is anisotropic or quasi-split, or has the Tits index ${}^3D_{4,1}^9$ or ${}^6D_{4,1}^9$.

Let the Tits index be ${}^3D_{4,1}^9$ or ${}^6D_{4,1}^9$. The anisotropic kernel G_{an} is isomorphic to $R_{R'/R}(\text{SL}_1(A))$ for some Azumaya algebra over R' with $\text{ind } A = \deg A = 2$. The Cartan matrix of D_4 shows that $\alpha_2 = 2\omega_2 - \omega_1 - \omega_3 - \omega_4$, so by Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_2) = -\text{cores}_{R'/R}(\beta_{\text{SL}_1(A)}(\omega_1)) = -\text{cores}_{R'/R}([A]).$$

□

Theorem 3 (${}^1\mathbf{E}_6$). *Every simple simply connected group G of inner type E_6 over R has one of the following Tits indices:*

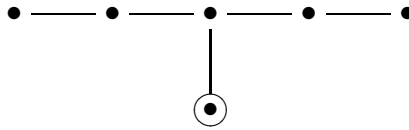


The possible anisotropic kernels are the following:

- simple simply connected anisotropic groups H of type D_4 over R with $\beta_H = 0$, in the case of ${}^1E_{6,2}^{28}$;
- $\mathrm{SL}_1(A)^2$, where A is an Azumaya algebra over R with $\mathrm{ind} A = \deg A = 3$, in the case of ${}^1E_{6,2}^{16}$.

In the case of ${}^1E_{6,0}^{78}$ G is anisotropic; in the case of ${}^1E_{6,6}^0$ G is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or the following:



Let us first exclude the latter case. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\mathrm{ind} A = \deg A = 6$. The Cartan matrix of E_6 shows that $\alpha_2 = 2\omega_2 - \omega_4$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_2) = -\beta_{\mathrm{SL}_1(A)}(\omega_3) = -3[A].$$

Hence $\exp A = 3$, but this contradicts Proposition 4.

In the case of ${}^1E_{6,2}^{28}$ the anisotropic kernel G_{an} is of type 1D_4 . We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5$, so

$$\begin{aligned} 0 &= \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_1); \\ 0 &= \beta_{G_{an}}(\alpha'_6) = -\beta_{G_{an}}(\omega_4). \end{aligned}$$

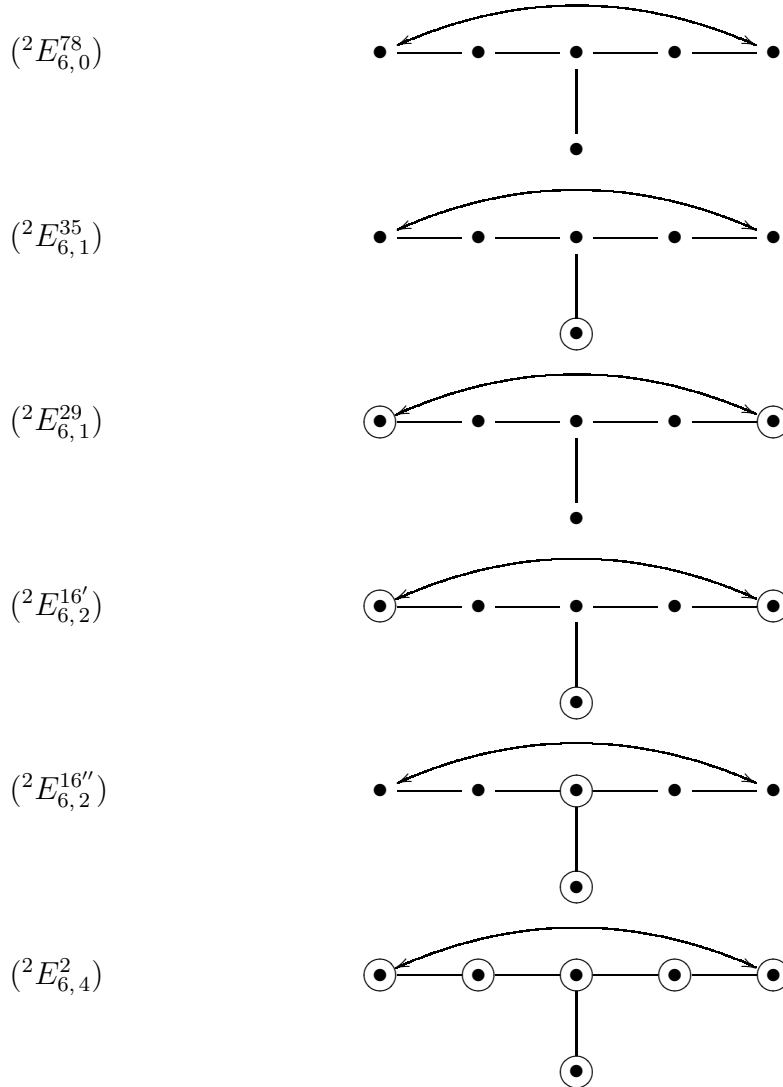
It follows that $\beta_{G_{an}} = 0$.

In the case of ${}^1E_{6,2}^{16}$ the anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2)$ for some Azumaya algebras A_1, A_2 over R with $\mathrm{ind} A_1 = \deg A_1 = \mathrm{ind} A_2 = \deg A_2 = 3$. We have $\alpha_4 = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G_{an}}(\alpha'_4) = \beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_2)}(\omega_1) = [A_1] - [A_2].$$

By Lemma 4 $A_1 \simeq A_2$. □

Theorem 3 (2E_6). *Every simple simply connected group G of type 2E_6 over R has one of the following Tits indices:*



Denote by $\mathrm{Spec} R'$ the orbit corresponding to $\{1, 6\}$ (so that R'/R is a connected quadratic étale extension). The possible anisotropic kernels are the following:

- simple simply connected anisotropic groups H of type 2A_5 over R with $\beta_H(\omega_3) = 0$, in the case of ${}^2E_{6,1}^{35}$;
- simple simply connected anisotropic groups H of type 2D_4 over R with $\beta_{H_{R'}}(\omega_3) = 0$, in the case of ${}^2E_{6,1}^{29}$;
- simple simply connected anisotropic groups H of type 2A_3 over R with $\beta_H(\omega_2) = 0$ and $\beta_{H_{R'}}(\omega_1) = 0$, in the case of ${}^2E_{6,2}^{16'}$;

- $R_{R'/R}(\mathrm{SL}_1(A))$, where A is an Azumaya algebra over R' with $\mathrm{ind} A = \deg A = 3$ and $\mathrm{cores}_{R'/R}([A]) = 0$, in the case ${}^2E_{6,2}^{16''}$.

In the case of ${}^2E_{6,0}^{78}$ G is anisotropic; in the case of ${}^2E_{6,4}^2$ G is quasi-split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is one of the listed above.

In the case of ${}^2E_{6,1}^{35}$ the anisotropic kernel G_{an} is a group of type 2A_5 . The Cartan matrix of E_6 shows that $\alpha_2 = 2\omega_2 - \omega_4$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_2) = -\beta_{G_{an}}(\omega_3).$$

In the case of ${}^2E_{6,1}^{29}$ the anisotropic kernel G_{an} is a group of type 2D_4 . Denote by $O = \mathrm{Spec} R'$ the orbit corresponding to $\{1, 6\}$. We have $\alpha_1 = 2\omega_1 - \omega_2$, so

$$0 = \beta_{G_{anO}}(\alpha'_O) = -\beta_{G_{anO}}(\omega_3).$$

In the case of ${}^2E_{6,2}^{16'}$ the anisotropic kernel G_{an} is a group of type 2A_3 . We have $\alpha_1 = 2\omega_1 - \omega_2$, $\alpha_2 = 2\omega_2 - \omega_4$, so

$$0 = \beta_{G_{anO}}(\alpha'_O) = -\beta_{G_{anO}}(\omega_1);$$

$$0 = \beta_{G_{an}}(\alpha'_4) = -\beta_{G_{an}}(\omega_2).$$

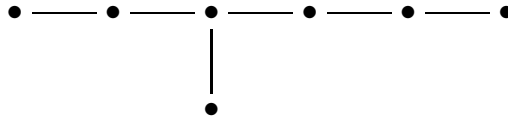
In the case of ${}^2E_{6,2}^{16''}$ the anisotropic kernel G_{an} is isomorphic to $R_{R'/R}(\mathrm{SL}_1(A))$, where A is an Azumaya algebra over R' with $\mathrm{ind} A = \deg A = 3$, $O \simeq \mathrm{Spec} R'$. We have $\alpha_4 = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, so

$$0 = \beta_{G_{an}}(\alpha'_4) = \mathrm{cores}_{R'/R}(\beta_{\mathrm{SL}_1(A)}(\omega_1)) = \mathrm{cores}_{R'/R}([A]).$$

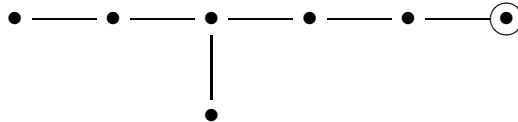
□

Theorem 3 (E₇). *Every simple simply connected group G of type E_7 over R has one of the following Tits indices:*

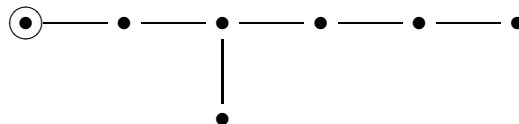
$(E_{7,0}^{133})$



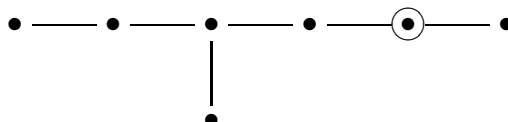
$(E_{7,1}^{78})$

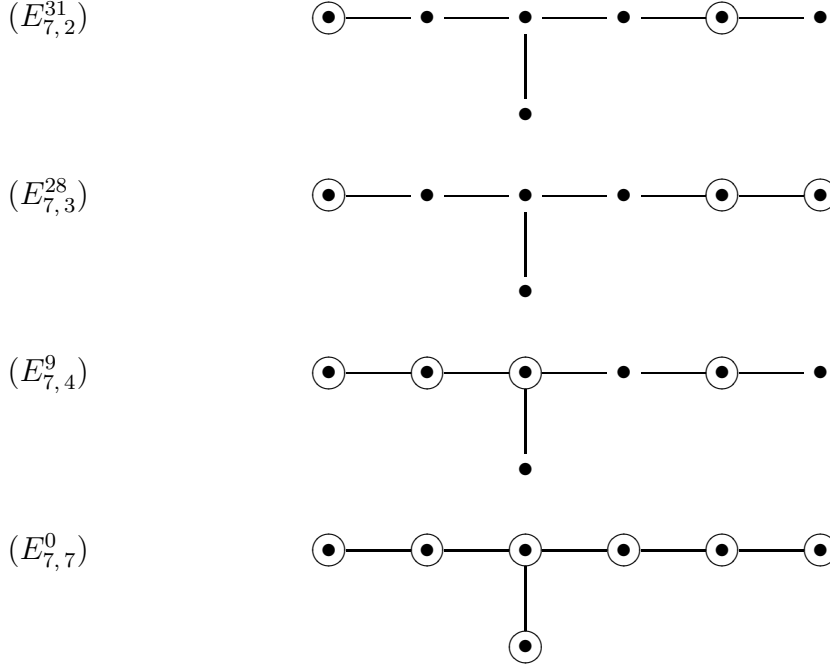


$(E_{7,1}^{66})$



$(E_{7,1}^{48})$



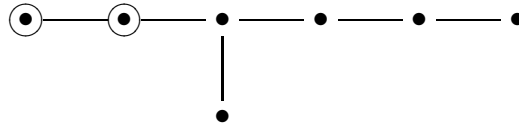


The possible anisotropic kernels are the following:

- simple simply connected anisotropic groups H of type 1E_6 over R with $\beta_H = 0$, in the case of $E_{7,1}^{78}$;
- simple simply connected anisotropic groups H of type 1D_6 over R with $\beta_H(\omega_5) = 0$, in the case of $E_{7,1}^{66}$;
- $H \times \mathrm{SL}_1(E)$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = 2$, H is a simple simply connected anisotropic group of type 1D_5 over R with $\beta_H(\omega_4) = [E]$, in the case of $E_{7,1}^{48}$;
- $H \times \mathrm{SL}_1(E)$, where E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = 2$, H is a simple simply connected anisotropic group of type 1D_4 over R with $\beta_H(\omega_1) = 0$ and $\beta_H(\omega_3) = [E]$, in the case of $E_{7,2}^{31}$;
- simple simply connected anisotropic groups H of type 1D_4 over R with $\beta_H = 0$, in the case of $E_{7,3}^{28}$;
- $\mathrm{SL}_1(A)^3$, where A is an Azumaya algebra over R with $\mathrm{ind} A = \deg A = 2$, in the case of $E_{7,4}^9$.

In the case of $E_{7,0}^{133}$ G is anisotropic; in the case of $E_{7,7}^0$ G is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or the following:



Let us first exclude the latter case. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\mathrm{ind} A = \deg A = 6$. The Cartan matrix of E_7 shows that $\alpha_3 = 2\omega_3 - \omega_1 - \omega_4$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_3) = -\beta_{\mathrm{SL}_1(A)}(\omega_2) = 2[A].$$

Hence $\exp A = 2$, but this contradicts Proposition 4.

In the case of $E_{7,1}^{78}$ the anisotropic kernel G_{an} is of type 1E_6 . We have $\alpha_7 = 2\omega_7 - \omega_6$, so

$$0 = \beta_{G_{an}}(\alpha'_7) = -\beta_{G_{an}}(\omega_6).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E_{7,1}^{66}$ the anisotropic kernel G_{an} is of type 1D_6 . We have $\alpha_1 = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_5).$$

In the case of $E_{7,1}^{48}$ the anisotropic kernel G_{an} is isomorphic to $H \times \mathrm{SL}_1(E)$, where H is a group of type 1D_6 , E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = 2$. We have $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_H(\omega_4) - \beta_{\mathrm{SL}_1(E)}(\omega_1) = -\beta_H(\omega_4) + [E].$$

In the case of $E_{7,2}^{31}$ the anisotropic kernel G_{an} is isomorphic to $H \times \mathrm{SL}_1(E)$, where H is a group of type 1D_4 , E is an Azumaya algebra over R with $\mathrm{ind} E = \deg E = 2$. We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_H(\omega_1);$$

$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_H(\omega_3) - \beta_{\mathrm{SL}_1(E)}(\omega_1) = -\beta_H(\omega_3) + [E].$$

In the case of $E_{7,3}^{28}$ the anisotropic kernel G_{an} is of type 1D_4 . We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_1);$$

$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E_{7,3}^{28}$ the anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A_1) \times \mathrm{SL}_1(A_2) \times \mathrm{SL}_1(A_3)$ for some Azumaya algebras A_1, A_2, A_3 over R with $\mathrm{ind} A_1 = \deg A_1 = \mathrm{ind} A_2 = \deg A_2 = \mathrm{ind} A_3 = \deg A_3 = 2$. We have $\alpha_4 = 2\omega_4 - \omega_2 - \omega_3 - \omega_5$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

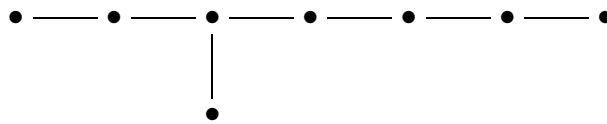
$$0 = \beta_{G_{an}}(\alpha'_4) = -\beta_{\mathrm{SL}_1(A_1)}(\omega_1) - \beta_{\mathrm{SL}_1(A_2)}(\omega_1) = [A_1] - [A_2];$$

$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_{\mathrm{SL}_1(A_2)}(\omega_1) - \beta_{\mathrm{SL}_1(A_3)}(\omega_1) = [A_2] - [A_3].$$

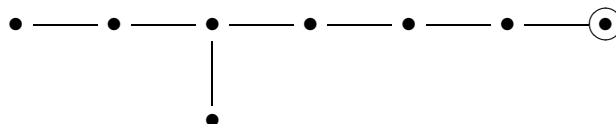
By Lemma 4 $A_1 \simeq A_2 \simeq A_3$. □

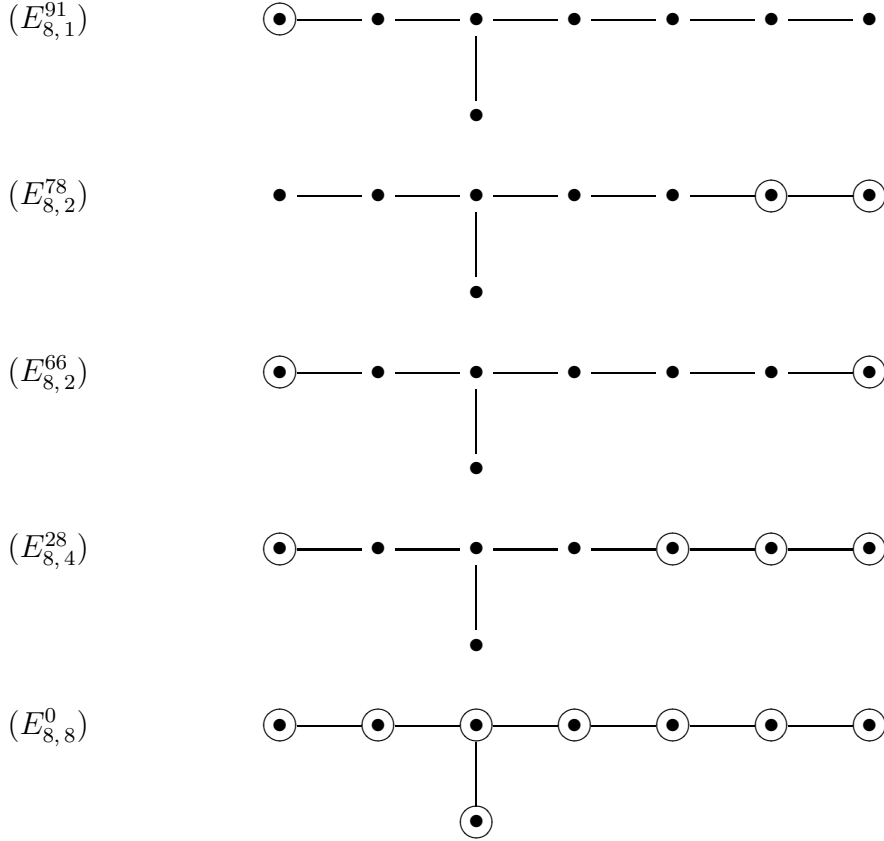
Theorem 3 (E₈). *Every simple simply connected group G of type E_8 over R has one of the following Tits indices:*

$(E_{8,0}^{248})$



$(E_{8,1}^{133})$





The possible anisotropic kernels are the following:

- simple simply connected anisotropic groups H of type E_7 over R with $\beta_H = 0$, in the case of $E_{8,1}^{133}$;
- simple simply connected anisotropic groups H of type 1D_7 over R with $\beta_H = 0$, in the case of $E_{8,1}^{91}$;
- simple simply connected anisotropic groups H of type 1E_6 over R with $\beta_H = 0$, in the case of $E_{8,2}^{78}$;
- simple simply connected anisotropic groups H of type 1D_6 over R with $\beta_H = 0$, in the case of $E_{8,2}^{66}$;
- simple simply connected anisotropic groups H of type 1D_4 over R with $\beta_H = 0$, in the case of $E_{8,4}^{28}$.

In the case of $E_{8,0}^{248}$ G is anisotropic; in the case of $E_{8,8}^0$ G is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is one of the listed above.

In the case of $E_{8,1}^{133}$ the anisotropic kernel G_{an} is of type E_7 . The Cartan matrix of E_8 shows that $\alpha_8 = 2\omega_8 - \omega_7$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_8) = -\beta_{G_{an}}(\omega_7).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E_{8,1}^{91}$ the anisotropic kernel G_{an} is of type 1D_7 . We have $\alpha_1 = 2\omega_1 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_6).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E_{8,2}^{78}$ the anisotropic kernel G_{an} is of type 1E_6 . We have $\alpha_7 = 2\omega_7 - \omega_6 - \omega_8$, so

$$0 = \beta_{G_{an}}(\alpha'_7) = -\beta_{G_{an}}(\omega_6).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E_{8,2}^{66}$ the anisotropic kernel G_{an} is of type 1D_6 . We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_8 = 2\omega_8 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_5);$$

$$0 = \beta_{G_{an}}(\alpha'_8) = -\beta_{G_{an}}(\omega_1).$$

It follows that $\beta_{G_{an}} = 0$.

In the case of $E_{8,4}^{28}$ the anisotropic kernel G_{an} is of type 1D_4 . We have $\alpha_1 = 2\omega_1 - \omega_3$, $\alpha_6 = 2\omega_6 - \omega_5 - \omega_7$, so

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_1);$$

$$0 = \beta_{G_{an}}(\alpha'_6) = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$. □

Theorem 3 (F_4). *Every simple simply connected group G of type F_4 over R has one of the following Tits indices:*

$$(F_{4,0}^{52}) \quad \bullet \text{ --- } \bullet \Longrightarrow \bullet \text{ --- } \bullet$$

$$(F_{4,1}^{21}) \quad \bullet \text{ --- } \bullet \Longrightarrow \bullet \text{ --- } \bigcirc$$

$$(F_{4,4}^0) \quad \bigcirc \text{ --- } \bigcirc \Longrightarrow \bigcirc \text{ --- } \bigcirc$$

The possible anisotropic kernels in the case of $F_{4,1}^{21}$ are simple simply connected anisotropic groups H of type B_3 over R with $\beta_H = 0$.

In the case of $F_{4,0}^{52}$ G is anisotropic; in the case of $F_{4,4}^0$ G is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or one of the following:

$$\bigcirc \text{ --- } \bullet \Longrightarrow \bullet \text{ --- } \bullet$$

$$\bigcirc \text{ --- } \bullet \Longrightarrow \bullet \text{ --- } \bigcirc$$

Let us exclude the two latter cases. In the first of them the anisotropic kernel G_{an} is of type C_3 . The Cartan matrix of F_4 shows that $\alpha_1 = 2\omega_1 - \omega_2$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_1) = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$, in contradiction with Proposition 5.

In the second case the anisotropic kernel G_{an} is of type C_2 . We have $\alpha_4 = 2\omega_4 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha'_4) = -\beta_{G_{an}}(\omega_1).$$

It follows that $\beta_{G_{an}} = 0$, in contradiction with Proposition 5.

In the case of $F_{4,1}^{21}$ G_{an} is of type B_3 . We have $\alpha_4 = 2\omega_4 - \omega_3$, so

$$0 = \beta_{G_{an}}(\alpha'_4) = -\beta_{G_{an}}(\omega_3).$$

It follows that $\beta_{G_{an}} = 0$. □

Theorem 3 (G_2). *Every simple simply connected group G of type G_2 over R has one of the following Tits indices:*

$$(G_{2,0}^{14}) \quad \bullet \longleftrightarrow \bullet$$

$$(G_{2,2}^0) \quad \circ \longleftrightarrow \circ$$

In the case of $G_{2,0}^{14}$ G is anisotropic; in the case of $G_{2,2}^0$ G is split.

Proof. By Lemma 3 and Proposition 3 the Tits index of G is either one of the listed above or the following:

$$\bullet \longleftrightarrow \circ$$

We need to exclude the latter case. The anisotropic kernel G_{an} is isomorphic to $\mathrm{SL}_1(A)$ for some Azumaya algebra A over R with $\mathrm{ind} A = \deg A = 2$. The Cartan matrix of G_2 shows that $\alpha_2 = 2\omega_2 - 3\omega_1$. By Theorem 2 we have

$$0 = \beta_{G_{an}}(\alpha'_2) = -3\beta_{\mathrm{SL}_1(A)}(\omega_1) = -3[A].$$

But by Proposition 4 $2[A] = 0$, hence $[A] = 0$, a contradiction. □

7. EXISTENCE OF INDICES

For the sake of completeness we give here a new uniform proof of the existence of indices of exceptional inner type over fields (note that all indices of outer types 2E_6 , 3D_4 , and 6D_4 appear already over number fields).

Theorem 4. *For any field F and any prescribed Tits index of exceptional inner type listed in Section 6, there exists a field extension E/F and a simple algebraic group G over E having that Tits index.*

Proof. Denote by H_0 the derived subgroup of the standard Levi subgroup of a parabolic subgroup P_0 in the split adjoint group G_0^{ad} over F . Now consider a *generic torsor* ζ under H_0 over an extension E/F . Recall that to construct ζ one chooses a faithful representation $H_0 \rightarrow \mathrm{GL}_n$, considers $E = F(\mathrm{GL}_n/H_0)$, and then takes the image in $H^1(E, H_0)$ of the generic point in $\mathrm{GL}_n/H_0(E)$ under the connecting map arising from the sequence

$$1 \rightarrow H_0 \rightarrow \mathrm{GL}_n \rightarrow \mathrm{GL}_n/H_0 \rightarrow 1.$$

After that we take the image ξ of ζ in $H^1(E, G_0^{ad})$ and consider the corresponding group G over E . Obviously G has a parabolic subgroup P whose Levi part is isomorphic to the group H corresponding to ζ . In general it may happen that H is isotropic, that is the Tits index of G contains more circled vertices than desired. Our goal is to show that if P_0 corresponds to one of the indices listed in Section 6, this is not the case.

To this end we employ an invariant $\text{cd}_p(X)$ of a projective homogeneous variety X called the *p-relative canonical dimension* of X ; see [6] for the definition and basic properties. We take X to be the variety of Borel subgroups of G . It is shown in [8, Proposition 6.1] that $\text{cd}_p(X)$ depends in an explicit monotonic way on a certain discrete invariant $J_p(G)$ of G (the *J-invariant*). By [8, Corollary 5.19] this invariant is the same for the group G itself and the derived subgroup H' of any parabolic subgroup of G . Also, if a group H corresponds to a generic torsor, $J_p(G)$ takes the maximal possible value, which is computed in [8, Example 4.7].

Now assume that the anisotropic kernel H' of G is less than H . From the one hand side, $\text{cd}_p(X)$ can be computed in terms of $J_p(H)$, which is known, since H is generic. From the other hand side, it can be computed in terms of $J_p(H')$, which does not exceed the known maximal possible value. If these values are distinct, we get a contradiction. Looking at the following table we see that for any two indices of the same type one can find p such that the maximal possible values of $\text{cd}_p(X)$ differ, and we are done.

Index	Maximal value of $\text{cd}_p(X)$	Index	Maximal value of $\text{cd}_p(X)$
${}^1E_{6,0}^{78}$	3, $p = 2$; 16, $p = 3$	$E_{8,0}^{248}$	60, $p = 2$; 28, $p = 3$; 24, $p = 5$
${}^1E_{6,2}^{28}$	3, $p = 2$	$E_{8,1}^{133}$	17, $p = 2$; 8, $p = 3$
${}^1E_{6,2}^{16}$	2, $p = 3$	$E_{8,1}^{91}$	14, $p = 2$
${}^1E_{6,6}^0$	0	$E_{8,2}^{78}$	3, $p = 2$; 8, $p = 3$
$E_{7,0}^{133}$	18, $p = 2$; 8, $p = 3$	$E_{8,2}^{66}$	8, $p = 2$
$E_{7,1}^{78}$	3, $p = 2$; 8, $p = 3$	$E_{8,4}^{28}$	3, $p = 2$
$E_{7,1}^{66}$	9, $p = 2$	$E_{8,8}^0$	0
$E_{7,1}^{48}$	10, $p = 2$	$F_{4,0}^{52}$	3, $p = 2$; 8, $p = 3$
$E_{7,2}^{31}$	6, $p = 2$	$F_{4,1}^{21}$	3, $p = 2$
$E_{7,3}^{28}$	3, $p = 2$	$F_{4,4}^0$	0
$E_{7,4}^9$	1, $p = 2$	$G_{2,0}^{14}$	3, $p = 2$
$E_{7,7}^0$	0	$G_{2,2}^0$	0

□

REFERENCES

- [1] N. Bourbaki, *Groupes et algèbres de Lie. Chapitres 4–6*, Hermann, Paris, 1968.
- [2] M. Demazure, A. Grothendieck, *Schémas en groupes*, Lecture Notes in Mathematics, Vol. 151–153, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
- [3] O. Gabber, *Some theorems on Azumaya algebras*, Lecture Notes in Mathematics, Vol. 844, Springer, Berlin-New York, 1981, 129–209.
- [4] R.S. Garibaldi, *Isotropic reductive algebraic groups*, J. of Algebra **210** (1998), 385–418.
- [5] J.C. Jantzen, *Darstellungen halbeinfacher Gruppen und kontravariante Formen*, J. Reine Angew. Math. **290** (1977), 117–141.
- [6] N. Karpenko, A. Merkurjev, *Canonical p-dimension of algebraic groups*, Advances in Math. **205** (2006), 410–433.
- [7] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Springer-Verlag, Berlin, 1991.
- [8] V. Petrov, N. Semenov, K. Zainoulline, *J-invariant of linear algebraic groups*, Ann. Sci. École Norm. Sup **41** (2008), 1021–1051.

- [9] M. Selbach, *Klassifikationstheorie halbeinfacher algebraischer Gruppen*, Bonner mathematische Schriften, Nr. 83, 1976.
- [10] T. Springer, *Linear algebraic groups*, 2nd ed., Birkhäuser, Boston, 1998.
- [11] J. Tits, *Classification of algebraic semisimple groups*, Algebraic groups and discontinuous subgroups, Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence RI, 1966, 33–62.
- [12] J. Tits. *Représentations linéaires irréductibles d’un groupe réductif sur un corps quelconque*, J. Reine Angew. Math. **247** (1971), 196–220.
- [13] J. Tits. *Strongly inner anisotropic forms of simple algebraic groups*, J. of Algebra **131** (1990), 648–677.
- [14] J. Tits, R.M. Weiss, *Moufang polygons*, Springer-Verlag, Berlin, 2002.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK, BONN, GERMANY
E-mail address: victorapetrov@googlemail.com

ST. PETERSBURG STATE UNIVERSITY, ST. PETERSBURG, RUSSIA, AND MATHEMATISCHES
INSTITUT, LUDWIG-MAXIMILIANS UNIVERSITÄT, MÜNCHEN, GERMANY
E-mail address: a_stavrova@mail.ru